Part I

Motivation

Introduction to Part I

In Part I we use the two smallest non-abelian finite simple groups, namely the alternating group A_5 and the general linear group $L_3(2)$ to define larger permutation groups of degrees 12 and 24, respectively. Specifically, we shall obtain highly symmetric sets of generators for each of the new groups and use these generating sets to deduce the groups' main properties. The first group will turn out to be the Mathieu group M_{12} of order $12 \times 11 \times 10 \times 9 \times 8 = 95\,040$ [70] and the second the Mathieu group M_{24} of order $24 \times 23 \times 22 \times 21 \times 20 \times 16 \times 3 = 244\,823\,040$ [71]; they will be shown to be quintuply transitive on 12 and 24 letters, respectively. These constructions were first described in refs. [31] and [32].

1

The Mathieu group M_{12}

1.1 The combinatorial approach

As is well known, the alternating group A_5 contains $4! = 24$ 5-cycles; these are all conjugate to one another in the symmetric group S_5, but, since 24 does not divide 60, they fall into two conjugacy classes of A_5 with 12 elements in each. Let $A \cong A_5$ act naturally on the set $Y = \{1, 2, 3, 4, 5\}$, and let $a = (1\ 2\ 3\ 4\ 5) \in A$ be one of these 5-cycles. Then the two classes may be taken to be

$$\Lambda = \{a^g \mid g \in A\} \quad \text{and} \quad \bar{\Lambda} = \{(a^2)^g \mid g \in A\}.$$

We shall define permutations of the set Λ, and eventually extend them to permutations of the set $\Lambda \cup \bar{\Lambda}$. In Table 1.1 we write the elements of Λ so that each begins with the number 1, and for convenience we label them using the projective line $P_1(11) = \{\infty, 0, 1, \ldots, \mathrm{X}\} = \{\infty\} \cup \mathbb{Z}_{11}$, where X stands for '10'. The other conjugacy class $\bar{\Lambda}$ is then labelled with the set $\{\bar{\infty}, \bar{0}, \bar{1}, \ldots, \bar{\mathrm{X}}\}$, with the convention that if $\lambda \in \Lambda$ is labelled n, then $\lambda^2 \in \bar{\Lambda}$ is labelled $\bar{n}$. Clearly, for $g \in A$, conjugation of elements of Λ by g yields a permutation of the 12 elements of Λ, and thus we obtain a transitive embedding of $A \cong A_5$ in the symmetric group S_{12}. Indeed, since A_5 is a simple group, it must be an embedding in the alternating group A_{12}.

We now define a new permutation of Λ, which we shall denote by s_1. It will be clear that permutations $s_2, \ldots, s_5$ can be defined similarly, by starting each of the 5-cycles in the definition of s_i with the symbol i. For $(1\ w\ x\ y\ z) \in \Lambda$ we define the following:

$$s_1 : (1\ w\ x\ y\ z) \mapsto (1\ w\ x\ y\ z)^{(x\ y\ z)} = (1\ w\ y\ z\ x).$$

We note that s_1 *is* a function from Λ to Λ; after all, the image of a given 5-cycle is certainly another 5-cycle and, since the permutation $(x\ y\ z)$ is even, it is in Λ rather than $\bar{\Lambda}$. Moreover, we see that s_1^3 acts as the identity

Table 1.1. Labelling of the 24 5-cycles with elements of the 12-point projective line

Λ				$\bar{\Lambda}$			
∞	(1 2 3 4 5)	0	(1 5 4 3 2)	$\bar{\infty}$	(1 3 5 2 4)	$\bar{0}$	(1 4 2 5 3)
1	(1 3 2 5 4)	2	(1 4 5 2 3)	$\bar{1}$	(1 2 4 3 5)	$\bar{2}$	(1 5 3 4 2)
9	(1 5 2 4 3)	7	(1 3 4 2 5)	$\bar{9}$	(1 2 3 5 4)	$\bar{7}$	(1 4 5 3 2)
4	(1 3 5 4 2)	8	(1 2 4 5 3)	$\bar{4}$	(1 5 2 3 4)	$\bar{8}$	(1 4 3 2 5)
3	(1 5 3 2 4)	6	(1 4 2 3 5)	$\bar{3}$	(1 3 4 5 2)	$\bar{6}$	(1 2 5 4 3)
5	(1 4 3 5 2)	X	(1 2 5 3 4)	$\bar{5}$	(1 3 2 4 5)	$\bar{X}$	(1 5 4 2 3)

on Λ, and so s_1 possesses an inverse (namely s_1^2) and is a permutation. But s_1 does not fix any 5-cycle, and so it has cycle shape 3^4 on Λ. It turns out that, if $\hat{a}$ denotes the image of a as a permutation of Λ, then $\hat{a}$ and s_1 generate a subgroup of A_{12} of order 95 040. In fact, we have the following:

$$\langle \hat{a}, s_1 \rangle = \langle s_1, s_2, s_3, s_4, s_5 \rangle \cong M_{12},$$

the Mathieu group [70], which was discovered in 1861. Explicitly, we see that

$$\hat{a} = (1\ 9\ 4\ 3\ 5)(2\ 7\ 8\ 6\ X) \text{ and } s_1 = (\infty\ 8\ X)(0\ 3\ 9)(1\ 4\ 7)(2\ 6\ 5).$$

It turns out that M_{12} is remarkable in that it can act non-permutation identically on two sets of 12 letters, and so acts intransitively on 24 letters with two orbits of length 12; it possesses an outer automorphism which can act on this set of 24 letters interchanging the two orbits. Not surprisingly, for us the two sets of size 12 will be Λ and $\bar{\Lambda}$. The element $\hat{a}$, by a slight abuse of notation, can be interpreted as an element of the alternating group A_{24} acting by conjugation on each of the two sets, with cycle shape 3^4 on each of them. Our new element s_1, however, requires a slight adjustment, and we define

$$\begin{aligned} s_1 : (1\ w\ x\ y\ z) &\mapsto (1\ w\ x\ y\ z)^{(x\ y\ z)} = (1\ w\ y\ z\ x) \text{ if } (1\ w\ x\ y\ z) \in \Lambda \\ &\mapsto (1\ w\ x\ y\ z)^{(z\ y\ x)} = (1\ w\ z\ x\ y) \text{ if } (1\ w\ x\ y\ z) \in \bar{\Lambda}. \end{aligned}$$

This yields

$$s_1 = (\infty\ 8\ X)(0\ 3\ 9)(1\ 4\ 7)(2\ 6\ 5)(\bar{\infty}\ \bar{3}\ \bar{5})(\bar{0}\ \bar{8}\ \bar{7})(\bar{1}\ \bar{6}\ \bar{9})(\bar{2}\ \bar{4}\ \bar{X}),$$

and in a similar way we obtain all five generators given in Table 1.2.

If we now define $S \cong S_5$ to be the set of all permutations of Y, then the odd permutations of S interchange the two sets Λ and $\bar{\Lambda}$ by conjugation. From the definition of the five elements $\{s_i \mid i = 1, \ldots, 5\}$, it is not surprising that conjugation by even elements of $\hat{S}$ simply permutes their subscripts in the natural way; however, odd elements of $\hat{S}$ permute *and invert*. These statements could be verified directly by conjugating the permutations given in Table 1.2, by generators for $\hat{A}$ and $\hat{S}$; however, we prefer to prove them formally. Thus we have Lemma 1.1.

Cambridge University Press & Assessment
978-0-521-85721-5 — Symmetric Generation of Groups: With Applications to many of the Sporadic Finite Simple Groups
Robert T. Curtis
Excerpt
More Information

Table 1.2. Action of the five symmetric generators of M_{12} on $\Lambda \cup \bar{\Lambda}$

$s_1 = (\infty\ 8\ X)(0\ 3\ 9)(1\ 4\ 7)(2\ 6\ 5)(\bar{\infty}\ \bar{3}\ \bar{5})(\bar{0}\ \bar{8}\ \bar{7})(\bar{1}\ \bar{6}\ \bar{9})(\bar{2}\ \bar{4}\ \bar{X})$
$s_2 = (\infty\ 6\ 2)(0\ 5\ 4)(9\ 3\ 8)(7\ X\ 1)(\bar{\infty}\ \bar{5}\ \bar{1})(\bar{0}\ \bar{6}\ \bar{8})(\bar{9}\ \bar{X}\ \bar{4})(\bar{7}\ \bar{3}\ \bar{2})$
$s_3 = (\infty\ X\ 7)(0\ 1\ 3)(4\ 5\ 6)(8\ 2\ 9)(\bar{\infty}\ \bar{1}\ \bar{9})(\bar{0}\ \bar{X}\ \bar{6})(\bar{4}\ \bar{2}\ \bar{3})(\bar{8}\ \bar{5}\ \bar{7})$
$s_4 = (\infty\ 2\ 8)(0\ 9\ 5)(3\ 1\ X)(6\ 7\ 4)(\bar{\infty}\ \bar{9}\ \bar{4})(\bar{0}\ \bar{2}\ \bar{X})(\bar{3}\ \bar{7}\ \bar{5})(\bar{6}\ \bar{1}\ \bar{8})$
$s_5 = (\infty\ 7\ 6)(0\ 4\ 1)(5\ 9\ 2)(X\ 8\ 3)(\bar{\infty}\ \bar{4}\ \bar{3})(\bar{0}\ \bar{7}\ \bar{2})(\bar{5}\ \bar{8}\ \bar{1})(\bar{X}\ \bar{9}\ \bar{6})$

Lemma 1.1 For s_i a permutation of $\Lambda \cup \bar{\Lambda}$ defined as above and $\pi \in S$, we have the following:

$$s_i^{\hat{\pi}} = s_{i^\pi} \text{ if } \pi \in A; \quad s_i^{\hat{\pi}} = s_{i^\pi}^{-1} \text{ if } \pi \in S \setminus A.$$

Proof Let $\lambda = (a_0\ a_1\ a_2\ a_3\ a_4) \in \Lambda$ and let $\pi \in A$. Then we have

$$\begin{aligned}
\lambda^{\hat{\pi}^{-1} s_j \hat{\pi}} &= (a_0^{\pi^{-1}}\ a_1^{\pi^{-1}}\ a_2^{\pi^{-1}}\ a_3^{\pi^{-1}}\ a_4^{\pi^{-1}})^{s_j \hat{\pi}} \\
&= (a_i^{\pi^{-1}}\ a_{i+1}^{\pi^{-1}}\ a_{i+2}^{\pi^{-1}}\ a_{i+3}^{\pi^{-1}}\ a_{i+4}^{\pi^{-1}})^{s_j \hat{\pi}} && (\text{where } j = a_i^{\pi^{-1}}) \\
&= (a_i^{\pi^{-1}}\ a_{i+1}^{\pi^{-1}}\ a_{i+3}^{\pi^{-1}}\ a_{i+4}^{\pi^{-1}}\ a_{i+2}^{\pi^{-1}})^{\hat{\pi}} && (\text{where } j^\pi = a_i) \\
&= (a_i\ a_{i+1}\ a_{i+3}\ a_{i+4}\ a_{i+2}) && (\text{where } j^\pi = a_i) \\
&= (a_i\ a_{i+1}\ a_{i+2}\ a_{i+3}\ a_{i+4})^{(a_{i+2}\ a_{i+3}\ a_{i+4})} && (\text{where } j^\pi = a_i) \\
&= \lambda^{s_{a_i}} = \lambda^{s_{j^\pi}}.
\end{aligned}$$

A similar calculation holds for $\lambda \in \bar{\Lambda}$, and so we have $s_j^{\hat{\pi}} = s_{j^\pi}$.

Further suppose that $\lambda = (a_0\ a_1\ a_2\ a_3\ a_4) \in \Lambda$ and let $\sigma \in S \setminus A$. Then we have

$$\begin{aligned}
\lambda^{\hat{\sigma}^{-1} s_j \hat{\sigma}} &= (a_0^{\sigma^{-1}}\ a_1^{\sigma^{-1}}\ a_2^{\sigma^{-1}}\ a_3^{\sigma^{-1}}\ a_4^{\sigma^{-1}})^{s_j \hat{\sigma}} \\
&= (a_i^{\sigma^{-1}}\ a_{i+1}^{\sigma^{-1}}\ a_{i+2}^{\sigma^{-1}}\ a_{i+3}^{\sigma^{-1}}\ a_{i+4}^{\sigma^{-1}})^{s_j \hat{\sigma}} && (\text{where } j = a_i^{\sigma^{-1}}) \\
&= (a_i^{\sigma^{-1}}\ a_{i+1}^{\sigma^{-1}}\ a_{i+4}^{\sigma^{-1}}\ a_{i+2}^{\sigma^{-1}}\ a_{i+3}^{\sigma^{-1}})^{\hat{\sigma}} && (\text{where } j^\sigma = a_i) \\
&= (a_i\ a_{i+1}\ a_{i+4}\ a_{i+2}\ a_{i+3}) && (\text{where } j^\sigma = a_i) \\
&= (a_i\ a_{i+1}\ a_{i+2}\ a_{i+3}\ a_{i+4})^{(a_{i+4}\ a_{i+3}\ a_{i+2})} && (\text{where } j^\sigma = a_i) \\
&= \lambda^{s_{a_i}^2} = \lambda^{(s_{j^\sigma})^{-1}},
\end{aligned}$$

where the third line follows because $\lambda^{\sigma^{-1}} \in \bar{\Lambda}$. As above, a similar calculation follows for $\lambda \in \bar{\Lambda}$, and so we have $s_j^{\hat{\sigma}} = (s_{j^\sigma})^{-1}$. □

The reader would be right to wonder why we chose to conjugate our 5-cycles $\lambda = (1\ w\ x\ y\ z)$ by the 3-cycle $(x\ y\ z)$ rather than by one of the other possibilities. In fact, we could have chosen any one of $(x\ y\ z)$, $(y\ z\ w)$, $(z\ w\ x)$ or $(w\ x\ y)$ and conjugated every element of Λ by it and every element of $\bar{\Lambda}$ by its inverse. In this way, we obtain four copies of the group M_{12} acting on $\Lambda \cup \bar{\Lambda}$, each of which contains the original group $\hat{A}$. A calculation involving normalizers, which is given explicitly in Section 1.3, shows that these are the only ways in which a copy of the alternating group A_5 acting transitively on 12 points can be extended to a copy of M_{12}.

In order better to understand the relationship between these four copies of M_{12}, it is useful to consider the normalizers of our groups $\hat{S}$ and $\hat{A}$ in the symmetric group Σ acting on the $12+12=24$ letters which $\hat{S}$ permutes. Now, the normalizer of $\hat{S}$ in Σ, factored by the centralizer of $\hat{S}$ in Σ, must be isomorphic to a subgroup of the automorphism group of S_5, which is just S_5 (since all automorphisms of S_5 are inner and its centre is trivial). Thus,

$$|N_\Sigma(\hat{S})| \leq |C_\Sigma(\hat{S})| \times 120;$$

so we wish to find all permutations of Σ which commute with $\hat{S}$. Before proceeding we recall the following elementary result.

Lemma 1.2 A permutation which commutes with a transitive group must be regular (i.e. has all its disjoint cycles of the same length), and a permutation which commutes with a doubly transitive group of degree greater than 2 must be trivial.

Proof Let $\pi \neq 1$ commute with a transitive group H. If π has cycles of differing length, then some non-trivial power of π possesses fixed points and, of course, commutes with H. But conjugation by H would then imply that every point must be fixed by this power of π, which is thus the identity. So we conclude that π could not have had cycles of differing lengths.

Suppose now that π commutes with the doubly transitive H and that $\pi : a_1 \mapsto a_2$ with $a_1 \neq a_2$. Choose $a_3 \notin \{a_1, a_2\}$. Then there exists a $\rho \in H$ with $a_1^\rho = a_1$ and $a_2^\rho = a_3$, and so $\pi = \pi^\rho : a_1 \mapsto a_3$. Thus we have a contradiction unless the degree is less than 3. □

Now, $\hat{S}$ acts transitively on $\Lambda \cup \bar{\Lambda}$, and so any permutation which commutes with it must be regular. Moreover, $\hat{S}$ has blocks of imprimitivity of size 4, namely the sets $\{\lambda, \lambda^2, \lambda^3, \lambda^4\}$, and it acts doubly transitively on these six blocks (as the projective general linear group $PGL_2(5)$). Thus a permutation centralizing $\hat{S}$ must fix each block, and there can be at most four such permutations. We now define

$$\tau : \lambda \mapsto \lambda^2 \quad \text{for } \lambda \in \Lambda \cup \bar{\Lambda}.$$

Clearly τ has order 4 and fixes each block. Moreover, we have

$$\lambda^{\hat{\sigma}\tau} = (\lambda^\sigma)^\tau = (\lambda^\sigma)^2 = (\lambda^2)^\sigma = (\lambda^\tau)^\sigma = \lambda^{(\tau\hat{\sigma})},$$

and so τ commutes with $\hat{S}$. We conclude that $C_\Sigma(\hat{S}) = \langle \tau \rangle$. We can now readily observe the following.

Lemma 1.3 Conjugation by the element τ cycles the four copies of M_{12} which extend $\hat{S}$ within Σ.

Proof For $\lambda = (1\ w\ x\ y\ z) \in \Lambda$, we have

$$\begin{aligned}(1\ w\ x\ y\ z)^{\tau^{-1}s_1\tau} &= [(1\ w\ x\ y\ z)^3]^{s_1\tau} = (1\ y\ w\ z\ x)^{s_1\tau} \\ &= (1\ y\ x\ w\ z)^\tau = (1\ x\ z\ y\ w) = (1\ w\ x\ y\ z)^{(w\ x\ z)},\end{aligned}$$

and similarly for $\lambda \in \bar{\Lambda}$. (Note that $\lambda^{\tau^{-1}} \in \bar{\Lambda}$.) Of course, this argument can be repeated for all four possible definitions of the generators s_i. □

For convenience, we give τ as a permutation of the 24 points of $\Lambda \cup \bar{\Lambda}$ as labelled in Table 1.1; thus

$$\tau = (\infty\ \bar{\infty}\ 0\ \bar{0})(1\ \bar{1}\ 2\ \bar{2})(9\ \bar{9}\ 7\ \bar{7})$$
$$(4\ \bar{4}\ 8\ \bar{8})(3\ \bar{3}\ 6\ \bar{6})(5\ \bar{5}\ \mathrm{X}\ \bar{\mathrm{X}}).$$

1.2 The regular dodecahedron

If we consider the group of rotational symmetries of the regular dodecahedron acting on its 12 faces, then the Orbit-Stabilizer Theorem soon tells us that the group has $12 \times 5 = 60$ elements. As we shall see later in this section, the 20 vertices of the dodecahedron fall (in two different ways) into five sets of four, each of which forms the vertices of a regular tetrahedron. These five tetrahedra are permuted by the group of rotational symmetries and all even permutations of them are realized; so the group is isomorphic to the alternating group A_5. Thus the transitive (but imprimitive) 12-point action of A_5 can be seen as rotational symmetries of the 12 faces. Before describing how our generators of order 3 appear acting on the faces, we show how a dodecahedron may be constructed from our group A.

For the sake of visual impact, we choose to replace the members of the set $Y = \{1, 2, 3, 4, 5\}$ by colours; thus, for example, we replace

1 by black,
2 by yellow,
3 by red,
4 by blue,
5 by green.

Now, for each of the 5-cycles $\lambda \in \Lambda$ we take a regular pentagon with its vertices coloured clockwise in the order in which the colours appear in λ. We now have a child's puzzle: can you piece these pentagons together, three at each vertex, so that the colours all match up? If we start with $\infty = (1\ 2\ 3\ 4\ 5)$, in the notation of Table 1.1, we have to ask which pentagon should be placed on its '23' edge. This must be $0 = (3\ 2\ 1\ 5\ 4)$, $1 = (3\ 2\ 5\ 4\ 1)$ or $5 = (3\ 2\ 4\ 1\ 5)$, but the first of these is clearly impossible as it would require a pentagon with two black vertices. Thus there are just two possibilities, and once that choice has been made the rest of the solution is forced. We thus obtain two dodecahedra with their 20 vertices labelled with five colours. Had we started with the 5-cycles of $\bar{\Lambda}$ rather than Λ, we should have obtained two more. In order to obtain generators for the usual version of M_{12} with the above labelling of the faces, we choose to place $1 = (3\ 2\ 5\ 4\ 1)$ on edge '23', and we obtain the solution shown in Figure 1.1.

Note that inverse elements λ and λ^{-1} correspond to opposite faces and that any two vertices having the same colour are the same distance apart.

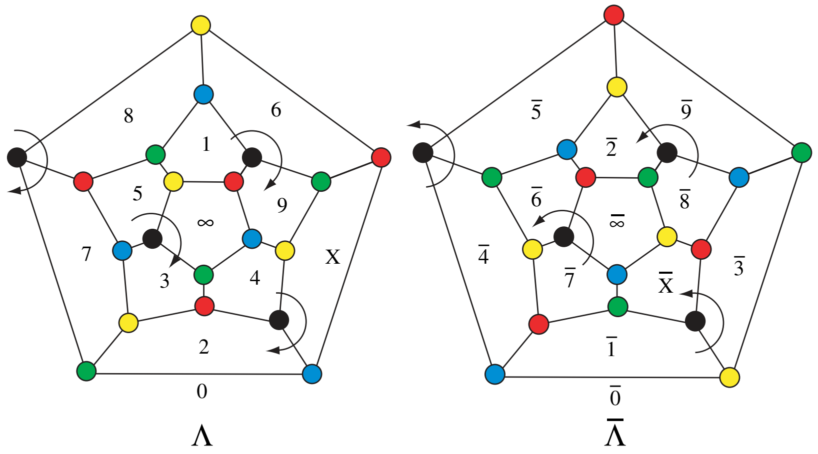

Figure 1.1. Two dodecahedra, each with its 20 vertices labelled using five colours.

In fact, if you move from any vertex along an edge, take the right fork at the first junction and the left fork at the second, then you will arrive at a vertex of the same colour. Thus the four vertices labelled with the same colour form the vertices of a regular tetrahedron, and we have partitioned the 20 vertices of the dodecahedron into five disjoint tetrahedra. There are in fact two such partitions and, had we chosen the option $5 = (3\ 2\ 4\ 1\ 5)$ instead, we should have obtained the other one. The two possible colourings furnished by $\bar{\Lambda}$ also correspond one each to these two partitions. Of course, A, the group of rotational symmetries of the dodecahedron, permutes these five tetrahedra in its natural action.

We are now in a position to read off the action of our 'black' generator s_1 on the dodecahedron shown in Figure 1.1. Recall that s_1 acts as $(\infty\ 8\ \mathrm{X})(0\ 3\ 9)(1\ 4\ 7)(2\ 6\ 5)$ on the faces. So we see that the rule is as follows.

> *Note that the three edges from a vertex lead to three faces; for each black vertex rotate these three faces clockwise.*

In order to see the outer automorphism of M_{12} and to appreciate the four possible sets of five symmetric generators of order 3, it is necessary to consider both the dodecahedra shown in Figure 1.1. Our canonical generator s_1 acts on both dodecahedra in the manner described in italics above, except that it 'twists' the latter in an anticlockwise sense. Note that the element τ, which cycles the four extensions, conjugates s_1 into

$$s_1^{\tau} = (\infty\ 4\ 9)(0\ 6\ \mathrm{X})(1\ 8\ 5)(2\ 3\ 7)$$
$$(\bar{\infty}\ \bar{8}\ \bar{\mathrm{X}})(\bar{0}\ \bar{3}\ \bar{9})(\bar{1}\ \bar{4}\ \bar{7})(\bar{2}\ \bar{6}\ \bar{5}),$$

which twists *not* the three faces joined by an edge to a vertex, but the three faces incident with a vertex. If we call the first type of twist a *deep* twist and the second type a *shallow* twist, then the four possible extensions of A to a copy of M_{12} are characterized as follows.

> *Choose one of the two partitions of the 20 vertices into five regular tetrahedra; then choose either deep twists or shallow twists.*

Thus one can see from Figure 1.1 that the conjugate generator s_1^τ corresponds to the other partition into disjoint tetrahedra and a shallow twist.

1.3 The algebraic approach

In this section we shall assume certain knowledge of the Mathieu group M_{12} as given in the ATLAS (see ref. [25], p. 33) and use this to show that the permutations produced in the preceding ways do indeed generate the group; in Section 1.4 we shall prove that our permutations generate a group with the familiar properties of M_{12} *without* assuming the existence of such a group.

Firstly note that $M \cong M_{12}$ contains a class of transitive subgroups isomorphic to the projective special linear group $L_2(11)$, and recall that this group contains (two classes) of transitive subgroups isomorphic to the alternating group A_5. Let A be such a subgroup and note that the stabilizer of a point in A is cyclic of order 5 and so subgroups of A isomorphic to A_4 act transitively, and so regularly, on the 12 points. Now, the normalizer in M_{12} of such an $H \cong A_4$ is a maximal subgroup of shape $A_4 \times S_3$, and so there is an element s_1 (of class $3B$ and cycle shape 3^4) commuting with H. Thus, under conjugation by A, s_1 will have five images which we may label $\{s_1, s_2, \ldots, s_5\}$. If $a \in A$ is chosen to have order 5, then we may choose our labels so that $s_i^a = s_{i+1}$, where $i = 1, \ldots, 4$ and $s_5^a = s_1$. The subgroup $\langle s_1, \ldots, s_5\rangle$ is normalized by A and, since the only proper subgroups properly containing A are isomorphic to $L_2(11)$ (as can be seen from the table of maximal subgroups in the ATLAS [25]) in which A_4 subgroups have trivial centralizer, we must have $\langle s_1, s_2, \ldots, s_5\rangle \cong M$.

Suppose now we start with a group $A \cong A_5$ acting transitively on 12 letters, and thus embedded in the symmetric group $\Sigma \cong S_{12}$. In order to obtain the configuration which we know exists in M_{12}, we must produce elements of order 3 which commute with subgroups of A isomorphic to A_4. Now, as above a subgroup $H \cong A_4$ of A must act regularly on the 12 points, since the point stabilizer in A is cyclic of order 5. We must seek the centralizer in Σ of H. But, as proved in Lemma 1.2, any permutation commuting with a transitive group must itself be regular, and so $C_\Sigma(H)$ has order at most 12. Moreover if H is realizing, say, the *left* regular representation of A_4, then it certainly commutes with the *right* regular representation. This is simply a consequence of the associativity of multiplication, for if L_x and R_y denote left multiplication by x and right multiplication by y, respectively, then

$$\begin{aligned} g(L_xR_y) &= (gL_x)R_y = (xg)R_y = (xg)y \\ &= x(gy) = (gy)L_x = (gR_y)L_x = g(R_yL_x), \end{aligned}$$

where g is any element of the group. Thus $C_\Sigma(H)$ is another copy of A_4 which contains precisely four cyclic subgroups of order 3. Conjugating these by the group A we obtain four sets of five generators. Certainly at least one of these sets must generate M_{12}, since we know that this configuration exists inside it. In order to see that each set generates a copy of M_{12}, we note that the normalizer in Σ of A has the following shape:

$$(2 \times \mathrm{A}_5)^{\cdot}2,$$

a slightly subtle group which contains no copy of S_5: every element in the outer half squares to the central involution times an element of A. The argument used in Section 1.1 applies, and we see that the normalizer of A in Σ factored by the centralizer must be isomorphic to a subgroup of S_5. But A acts imprimitively on the 12 letters with blocks of size 2, and acts doubly transitively (as $\mathrm{L}_2(5)$) on the six blocks. Thus the only non-trivial element of Σ centralizing A is an element of order 2 interchanging each of the pairs which constitute the blocks. This, of course, corresponds to the central reflection of the dodecahedron which interchanges opposite faces. This shows that the normalizer has maximal order $(2 \times 5!)$. We can obtain it by adjoining τ times an odd permutation of $\hat{S}$ to our group $\hat{A}$ of Section 1.1. Thus,

$$\tau(4\ 5) = (\infty\ 7\ 0\ 9)(1\ \mathrm{X}\ 2\ 5)(3\ 4\ 6\ 8)$$

can be readily checked to normalize our $\hat{A}$, which acts as rotational symmetries of the dodecahedron and is generated by

$$\hat{A} = \langle (1\ 9\ 4\ 3\ 5)(2\ 7\ 8\ 6\ \mathrm{X}), (\infty\ 1)(7\ \mathrm{X})(0\ 2)(9\ 5)(3\ 6)(4\ 8) \rangle.$$

The four sets of five generators are conjugate under the action of the above element of order 4 and, since one set at least had to generate a copy of M_{12}, they all do. We note that, if the four copies of M_{12} containing our initial A_5 are placed at the vertices of a square so that conjugation by the element $\tau(4\,5)$ above rotates the square through 90°, then adjacent copies intersect in subgroups isomorphic to $\mathrm{L}_2(11)$, while diagonally opposite copies intersect in just the initial A_5.

1.4 Independent proofs

In this section we define $M = \langle s_1, s_2, \ldots, s_5 \rangle$ to be the subgroup of Σ, the symmetric group on 12 letters, generated by the s_i as defined in the unbarred part of Table 1.2, and deduce the well known properties of M_{12}. Thus the s_i are as displayed in Table 1.3 and

$$\hat{a} = (1\ 9\ 4\ 3\ 5)(2\ 7\ 8\ 6\ \mathrm{X}).$$

Firstly we show the following lemma.