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A Practical Guide to the Invariant Calculus

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CAMBRIDGE
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University Printing House, Cambridge CB2 8BS, United Kingdom

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477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

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www.cambridge.org
Information on this title: www.cambridge.org/9780521857017

First published 2010

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-85701-7 Hardback

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Preface

I first became enamoured of the Fels and Olver formulation of the moving frames theory when it helped me solve a problem I had been thinking about for several years. I set about reading their two 50-page papers, and made a 20-page handwritten glossary of definitions. I was lucky in that I was able to ask Peter Olver many questions and am eternally grateful for the answers.

I set about solving the problems that interested me, and realised there were so many of them that I could write a book. I also wanted to share my amazement at just how powerful the methods were, and at the essential simplicity of the central idea. What I have tried to achieve in this book is a discussion rich in examples, exercises and explanations that is largely accessible to a graduate student, although access to a professional mathematician will be required for some parts. I was extremely fortunate to have six students read through various drafts from the very beginning. The comments and hints they needed have been incorporated, and I have not hesitated to put in a discussion, example, exercise or hint that might be superfluous to a professional.

There is a fair amount of original material in this book. Even though some of the problems addressed here have been solved using moving frames already, I have re-proved some results to keep both solution methods and proofs within the domain of the mathematics developed here. I love coming up with simpler solutions. In particular, the variational methods developed in Chapter 7 are my own. The theorem on moving frames and Noether's Theorem, which was discovered and proved with Tania Gonçalves, particularly pleases me. The application of moving frames to the solution of invariant ordinary differential equations is also new. I was particularly chuffed to solve the Chazy equation using relatively simple calculations, see Chapter 6. Theorem 5.2.4 allowing one to write down the curvature matrices in terms of a matrix representation of the frame was published earlier in Mansfield and van der Kamp (2006), and there are some fun exercises giving new applications. Finally some minor (and not so minor) errors in the original papers have been corrected.

The natural setting of the problems that interested me did not fit well with the language of differential geometry in which all discussions of moving frames were couched, so I set about casting the calculations into ordinary undergraduate calculus in order to explain it in my papers and then to teach it to my students. It was clear that a major benefit of Fels and Olver's formulation of the central concept was that it actually freed the moving frame method from the confines of differential geometry; that it could apply equally well to differential difference problems, to discrete problems, to all kinds of numerical approximations and so on. In any event, there are serious problems with that language as an expository tool.† Thus when I decided to write up my notes into a book, I was clear in my own mind that I was not going to use the exterior calculus as the primary expository language. Nevertheless, it is important to have available coordinate-free expressions if we are not to suffer 'death by indices'. What I wanted was a language that offered concrete models of objects like smooth functions, vectors and vector fields, capable of use in both finite and infinite dimensional spaces, that was linked in an open, explicit and well-defined way to multivariable calculus, and for which there was a good literature where the central significant theorems were proved properly. The language I needed, and use, is that of Differential Topology. I learned this subject twice, first at the University of Sydney in lectures given by M. J. Field, and then at the University of Wisconsin, Madison, in a year long course given by Dennis Stowe. I am extremely grateful to them both. The notation and language that I use in this book is what they both independently taught me, which has stood me in good stead my whole career.

A huge contribution to the theory of moving frames, as they can be studied rigorously in a symbolic computation environment, has been made by Evelyne Hubert. One of the main benefits of the Fels and Olver formulation of moving frames is that much of the calculation can be done symbolically in a computer algebra environment. The fact that one can have a symbolic calculus of invariants, without actually solving for the frame, is what turns this theory from the merely beautiful to the both beautiful and useful; this is the hallmark of the best mathematics. From the point of view of *rigorous* symbolic computation, though, there were problems, in particular with the need to invoke the implicit function theorem because this is a non-constructive step. Evelyne Hubert and Irina Kogan (Hubert and Kogan, 2007a) provide algebraic foundations to the moving frame method for the construction of local invariants and present a

† Don't get me started.

parallel algebraic construction that produces algebraic invariants together with the relations they satisfy. They then show that the algebraic setting offers a computational solution to the original differential geometric construction.

A second problem solved by Evelyne Hubert was the lack of a theory to analyse the differential systems resulting from invariantisation, since these involve non-commuting differential operators. Indeed, none of the edifice of mathematics that had been produced to study over determined differential systems rigorously was applicable, although an equivalent theory was needed for the applications (Mansfield, 2001). In a beautiful exposition (Hubert, 2005), the web of difficulties was pulled apart, the necessary concepts and results were lined up in order, and the required theory was developed.

A third problem solved by Hubert was that of proving that a certain small, finite set of syzygies, or differential relations satisfied by the invariants, generated the complete set of syzygies (Hubert, 2009a). This was important since the theorem written down by Fels and Olver turned out to be false in general.

Finally, Hubert finds a set of generators of the algebra of differential invariants that are not only simple to calculate but simple to conceptualise (Hubert, 2009b).

To give an exposition of these papers at the level I wrote this volume would require another volume, with a substantial expository section on over determined systems. However, the papers are accessible and I commend them to the reader.

When I started to view the material from the point of view of my target audience, primarily people wanting to use the methods but not having learnt (nor wanting to learn) Differential Geometry, and also graduate students, I came to realise that the subject involves a significant range of mathematics that could not realistically be assumed knowledge. Brief but necessary remarks on topics from transversality to foliations to jet bundles, and on calculations in Lie algebras and the variational calculus, all swelled to much longer expository sections than I anticipated. One central classical theorem for which I could not find a decent modern exposition of the proof was Frobenius' Theorem, so I have outlined the proof in a series of exercises. The outline is based on that given in lectures at the University of Wisconsin, Madison, by Dennis Stowe, to whom I acknowledge my debt.

In writing this book I have tried to steer a course through the material that is both honest and pragmatic. If being rigorous would have involved too long a detour, I chose computation of examples and discussion over rigour; it is more insightful to discuss the meaning rather than the proof of a result when there is a good text that can be consulted for further reading. Where I do give a proof, though, I aimed for the proof to follow rigorously from the established base of knowledge. Interestingly, sometimes not even the cleanest, simplest proofs reveal the inner truth: the full understanding of theorems can only be achieved after a range of examples can be computed. I give many exercises, hints, and details in my own calculations to help my readers to two levels of computational expertise: first, to be able to correctly work simple examples that can be done by hand or performed interactively with a computer algebra package, and second, to be able to write a computer program to do his or her own larger examples.

I wish to thank Peter Olver, Evelyne Hubert, Peter Hydon and Francis Valiquette, who sent me comments. I had some great discussions with Gloria Marí Beffa, resulting in several beautiful examples that are described in the text. Peter van der Kamp's insistence on in-depth detail for his own understanding of moving frames made this a much better book. Tania Gonçalves, Richard Hoddinott, Jun Zhao and Andrew Wheeler worked through the exercises; readers can thank them for the hints and for amplified discussions in various places. I road tested the very first set of notes on Emma Berry and Andrew Martin whose comments helped me see things from my target audience's point of view.

As ever, I wish to thank my dear husband Peter Clarkson who supported me in a million different ways when the going got tough. I have faced and overcome some extraordinary obstacles in order to have a mathematical career; I have my father Dr Colin Mansfield, my PhD thesis supervisor Dr Edward Fackerell (Sydney), and my mentor Professor Arieh Iserles (Cambridge) to thank for their extraordinary timely support. Words cannot express how lucky and how grateful I feel to have such stalwart friends and fellow travellers.

The author would like to acknowledge the Engineering and Physical Sciences Research Council (UK) grant, 'Symmetric variational problems' EP/E001823/1.