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Introduction

In the mountains the shortest way is from peak to peak, but for that route thou must have long legs.

F. Nietzsche, *Thus Spake Zarathustra*

Multicarrier (MC) modulations such as orthogonal frequency division multiplexing (OFDM) and discrete multitone (DMT) are efficient technologies for the implementation of wireless and wireline communication systems. Advantages of MC systems over single-carrier ones explain their broad acceptance for various telecommunication standards (e.g., ADSL, VDSL, DAB, DVB, WLAN, WMAN). Yet many more appearances are envisioned for MC technology in the standards to come. A relatively simple implementation is possible for MC systems. Low complexity is due to the use of fast discrete Fourier transform (DFT), avoiding complicated equalization algorithms. Efficient performance of MC modulation is especially vivid in channels with frequency selective fading and multipath. Nonetheless, still a major barrier for implementing MC schemes in low-cost applications is its nonconstant signal envelope, making the transmission sensitive to nonlinear devices in the communication path. Amplifiers and digital-to-analog converters distort the transmit signals leading to increased symbol error rates, spectral regrowth, and reduced power efficiency compared with single carrier systems. Naturally, the transmit signals should be restricted to those that do not cause the undesired distortions. A reasonable measure of the relevance of the signals is the ratio between the peak power values to their average power (PAPR). Thus the goal of peak power control is to diminish the influence of transmit signals with high PAPR on the performance of the transmission system. Alternatives are either the complete exclusion of such signals or an essential decrease in the probability of their appearance. Neither of these goals can be achieved without a decrease in the efficient transmission rate or performance penalty.

In this monograph I describe methods of analysis and control of peak power effects on the performance of MC communication systems. This includes analysis of statistical properties of peak distributions in MC signals, descriptions of MC signals with low peaks, and approaches to decreasing high peaks in transmitted signals. Consequently, the organization of the book is as follows. In Chapter 2, I provide general definitions related to MC communication systems and MC signals, and introduce the main definitions related to peaks of MC signals. This is followed by a description of nonlinearities in power amplifiers and their influence on the performance. In Chapter 3, necessary mathematical tools are described. This is necessary since the mathematical arsenal of the peak power control research consists of many seemingly unrelated methods. Among them are harmonic analysis, probability, algebra, combinatorics, and coding theory. In Chapter 4, I explain how the continuous problem of peak estimation can be reduced to the discrete problem of analysis of maxima in the sampled MC signal. Chapter 5 deals with statistical distribution of peaks in MC signals. It is shown that the peak distribution is concentrated around a typical value and the proportion of signals that are essentially different from the typical maximum of the absolute value is small. Chapter 6 extends the analysis of the previous chapter to MC signals defined by coded information. In Chapter 7, I describe methods to construct MC signals with much smaller peaks than is typical. Finally, in Chapter 8, I analyze approaches for decreasing peaks in MC signals. Several algorithms are introduced and compared. Notes in the end of each chapter provide historical comments and attribute the results appearing in the chapter.

Several related topics are not treated in this monograph. For peak power control in CDMA see, e.g., [43, 64, 118, 228, 230, 231, 285, 299, 300, 308, 309, 324, 325, 326, 363, 421, 422, 423, 445] and references therein. Peak power reduction in MIMO systems is discussed in, e.g., [1, 66, 67, 154, 234, 235, 241, 278, 338, 389, 395, 411, 456]. For analysis of peak power control in OFDMA see, e.g., [154, 315, 427, 453]. Aspects of peak power reduction in radar systems are considered in [55, 236, 237, 238, 279, 430]. Peak power control in optical signals is considered in [371, 375].

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Multicarrier signals

In this chapter, I introduce the main issues we will deal with in the book. In Section 2.1, I describe a multicarrier (MC) communication system. I introduce the main stages that the signals undergo in MC systems and summarize advantages and drawbacks of this technology. Section 2.2 deals with formal definitions of the main notions related to peak power: peak-to-average power ratio, peak-to-mean envelope power ratio, and crest factor. In Section 2.3, I quantify the efficiency of power amplifiers and its dependence on the power of processed MC signals. Section 2.4 introduces nonlinear characteristics of power amplifiers and describes their influence on the performance of communication systems.

2.1 Model of multicarrier communication system

The basic concept behind multicarrier (MC) transmission is in dividing the available spectrum into subchannels, assigning a carrier to each of them, and distributing the information stream between subcarriers. Each carrier is modulated separately, and the superposition of the modulated signals is transmitted. Such a scheme has several benefits: if the subcarrier spacing is small enough, each subchannel exhibits a flat frequency response, thus making frequency-domain equalization easier. Each substream has a low bit rate, which means that the symbol has a considerable duration; this makes it less sensitive to impulse noise. When the number of subcarriers increases for properly chosen modulating functions, the spectrum approaches a rectangular shape. The multicarrier scheme shows a good modularity. For instance, the subcarriers exhibiting a disadvantageous *signal-to-noise ratio* (SNR) can be discarded. Moreover, it is possible to choose the constellation size (bit loading) and energy for each subcarrier, thus approaching the theoretical capacity of the channel.

Figure 2.1 presents the structure of a MC transmitter. Let n be the number of subcarriers in this system. The following processing stages are employed to derive the transmit signal. Redundancy defined by an error-correcting code is appended to

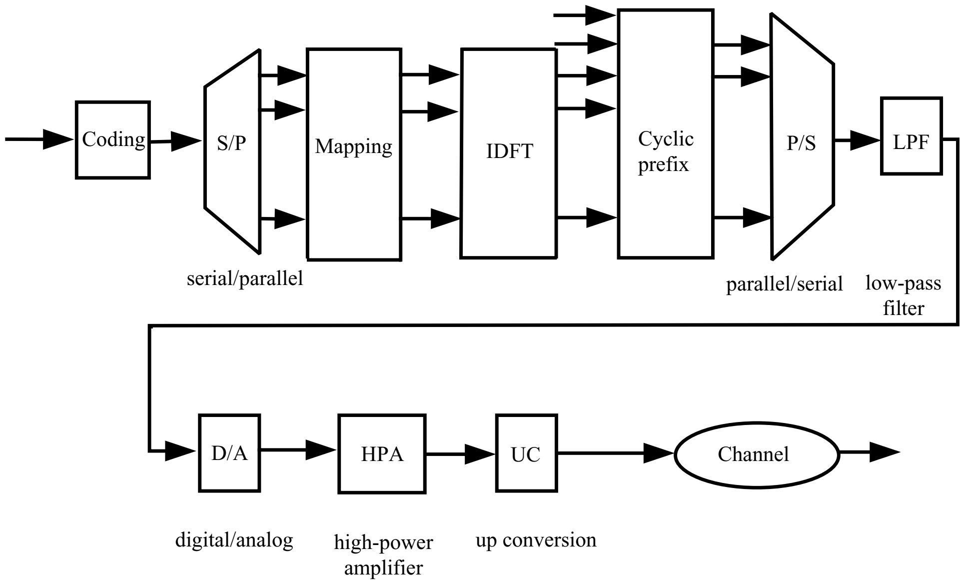

Figure 2.1 MC transmitter

the input information. The encoded data is converted to parallel form, and is mapped to n complex numbers defining points in the constellation used for modulation (e.g., QAM or PSK). These n complex numbers are inserted into an inverse discrete Fourier transform (IDFT) block, which outputs the time equidistributed samples of the baseband signal. The next block introduces a *guard interval* (GI) intended for diminishing the effect of the delay of the multipath propagation. The guard interval is usually implemented as a cyclic prefix (CP). Because of the CP, the transmit signal becomes periodic, and the effect of the time-dispersive multipath channel becomes equivalent to a cyclic convolution, discarding the GI at the receiver. Thus the effect of the multipath channel is limited to a pointwise multiplication of the transmitted data constellations by the channel transfer function, that is, the subcarriers remain orthogonal. Being converted back to the serial form, the samples are transformed by a low-pass filter (LPF) to give a continuous signal. This continuous signal is amplified by a high-power amplifier (HPA). Finally, if necessary, the baseband signal becomes passband by translation to a higher frequency. The reverse steps are performed by the receiver.

Implementation advantages of the MC communication system come from the simple structure of the DFT, which can be realized with a complexity proportional to $n \ln n$. Also, the equalization required for detecting the data constellations is an elementwise multiplication of the DFT output by the inverse of the estimated channel transfer function.

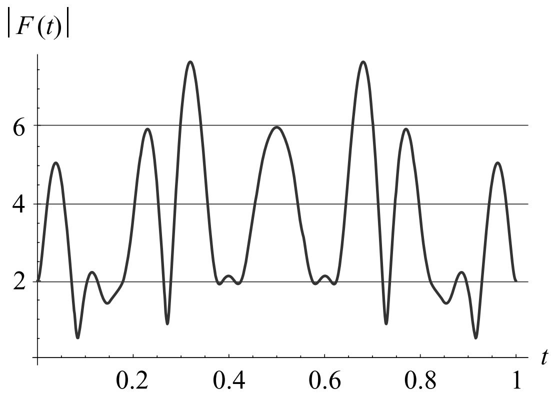

Figure 2.2 Envelope of a BPSK modulated MC signal for $n = 16$

However, several disadvantages arise with this concept, the most severe of which is the highly nonconstant envelope of the transmit signal (see Fig. 2.2), making MC modulation very sensitive to nonlinear components in the transmission path. A key component is the high-power amplifier (HPA). Owing to cost, design, and, most importantly, power efficiency considerations, the HPA cannot resolve the dynamics of the transmit signal and inevitably cuts off the signal at some point, causing additional in-band distortion and adjacent channel interference. The power efficiency penalty is certainly the major obstacle to implementing MC systems in low-cost applications. Moreover, in power-limited regimes determined by regulatory bodies, the average power is reduced in comparison to single-carrier systems, in turn reducing the range of transmission.

The main goal of peak power control is to diminish the influence of high peaks in transmit signals on the performance of the transmission system.

2.2 Peak power definitions

Let me give a more detailed description of the signals in the MC communication system. Denote by n the number of subcarriers (tones). The system receives at each time instant $0, \Upsilon, 2\Upsilon, \ldots$ a collection of n constellation symbols a_k, $k = 0, \ldots, n-1$, where $a_k \in \mathbb{C}$, carrying the information to be transmitted. The subset $\mathcal{Q}$ of possible values of a_k depends on the type of the carrier modulation. The most popular complex constellations are BPSK, M-QAM, and M-PSK. We assume

$$\text{BPSK} = \{-1, 1\},$$

$$M\text{-QAM} = \left\{A\left((2m_1 - 1) + \imath\,(2m_2 - 1)\right), m_1, m_2 \in \left\{-\frac{m}{2} + 1, \ldots, \frac{m}{2}\right\}\right\},$$

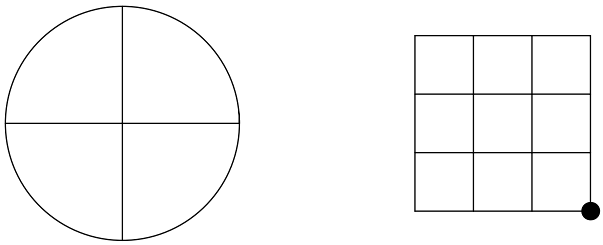

Figure 2.3 Examples of standard constellations

for natural numbers $m > 1$, M such that $M = m^2$ and $A^2 = \frac{3}{2}(M-1)$, and

$$M\text{-PSK} = \left\{1, e^{\frac{2\pi\imath}{M}}, \ldots, e^{\frac{2\pi\imath(M-1)}{M}}\right\}$$

for $M > 2$ and $\imath = \sqrt{-1}$. With such normalization, the average energy of a constellation point is 1. Notice that the (envelope) power of all M-PSK signals is the same and equals n. Another example of signals with equal power is provided by *spherical constellations* for which the only imposed restriction is that $\sum_{k=0}^{n-1} |a_k|^2 = n$. However, e.g., for MC signals using M-QAM, with $M > 4$, there are signals having different power. In the case of constellation points of varying energy, we scale the signal in such a way that the average energy is normalized to 1,

$$E_{av} = E_{av}(\mathcal{Q}) = \frac{1}{|\mathcal{Q}|} \sum_{a \in \mathcal{Q}} |a|^2 = 1. \tag{2.1}$$

We denote the maximum energy of a constellation point by $E_{max} = E_{max}(\mathcal{Q})$, where

$$E_{max} = \max_{a \in \mathcal{Q}} |a|^2. \tag{2.2}$$

Let f_0 be the carrier frequency and f_s be the bandwidth of each tone. Ignoring the possibility of assigning a guard time (which is a common assumption) we set $f_s = \frac{1}{\Upsilon}$. The transmitted signal on the interval $t = \left[0, \frac{1}{f_s}\right)$ is then represented by the real part of

$$S_{\mathbf{a}}(t) = \sum_{k=0}^{n-1} a_k e^{2\pi\imath(f_0 + kf_s)t}. \tag{2.3}$$

The *instantaneous power* of the transmit signal is $(\Re(S_{\mathbf{a}}(t)))^2$, while $|S_{\mathbf{a}}(t)|^2$ is called the *envelope power*. Denoting $\zeta = \frac{f_0}{f_s}$ and considering the signal on the interval $[0, 1)$, we have the following definition for the *peak-to-average power* *ratio* of $S_{\mathbf{a}}(t)$:

$$\mathtt{PAPR}(\mathbf{a}) = \frac{1}{\sum_{k=0}^{n-1} |a_k|^2} \cdot \max_{t\in[0,1)} \left| \Re \left(\sum_{k=0}^{n-1} a_k \mathrm{e}^{2\pi\imath(\zeta+k)t} \right) \right|^2 . \tag{2.4}$$

It is straightforward that

$$\mathtt{PAPR}(\mathbf{a}) \le \mathtt{PMEPR}(\mathbf{a}) = \frac{1}{\sum_{k=0}^{n-1} |a_k|^2} \cdot \max_{t\in[0,1)} |F_{\mathbf{a}}(t)|^2 , \tag{2.5}$$

where

$$F_{\mathbf{a}}(t) = \sum_{k=0}^{n-1} a_k \mathrm{e}^{2\pi\imath kt}, \tag{2.6}$$

and PMEPR stands for the *peak-to-mean-envelope-power ratio*. Another often considered parameter is the *crest factor* (CF) which is just the square root of PMEPR,

$$\mathtt{CF}(\mathbf{a}) = \sqrt{\mathtt{PMEPR}(\mathbf{a})}. \tag{2.7}$$

Although PMEPR provides an upper bound on PAPR, it is quite accurate for big values of ζ. Indeed, in the definition of PAPR we use $(\Re(S_{\mathbf{a}}(t)))^2$, while $|\mathrm{e}^{2\pi\imath\zeta t} S_{\mathbf{a}}(t)|^2$ is used for PMEPR. If ζ is large, a tiny change in t drastically modifies the phase of $\mathrm{e}^{2\pi\imath\zeta t} S_{\mathbf{a}}(t)$ while the value and phase of band-limited $S_{\mathbf{a}}(t)$ does not change significantly. Thus just in the very close vicinity of t_0 in which the maximum of $|S_{\mathbf{a}}(t)|$ is attained, it is possible to find t_1 such that $\mathrm{e}^{2\pi\imath\zeta t_1} S_{\mathbf{a}}(t_1)$ is real and $|S_{\mathbf{a}}(t_1)|$ is very close to $|S_{\mathbf{a}}(t_0)|$. This analysis will be quantified later in Section 4.6.

In what follows, we will often consider a situation when the vector $\mathbf{a}$ belongs to a discrete set $\mathcal{C} \subset \mathbb{C}^n$, of size $|\mathcal{C}|$, called *code*. In this context, assuming that all the code words are equiprobable, we define the average power of an MC signal from code $\mathcal{C}$ as

$$P_{av}(\mathcal{C}) = \frac{1}{|\mathcal{C}|} \sum_{\mathbf{a}\in\mathcal{C}} \sum_{k=0}^{n-1} |a_k|^2. \tag{2.8}$$

Consequently,

$$\mathtt{PAPR}(\mathcal{C}) = \frac{1}{P_{av}(\mathcal{C})} \cdot \min_{\mathbf{a}\in\mathcal{C}} \max_{t\in[0,1)} |\Re\left(S_{\mathbf{a}}(t)\right)|^2 , \tag{2.9}$$

where $S_{\mathbf{a}}(t)$ is defined in (2.3), and

$$\mathtt{PMEPR}(\mathcal{C}) = \frac{1}{P_{av}(\mathcal{C})} \cdot \min_{\mathbf{a}\in\mathcal{C}} \max_{t\in[0,1)} |F_{\mathbf{a}}(t)|^2 , \tag{2.10}$$

where $F_{\mathbf{a}}(t)$ is defined in (2.6). Finally,

$$\mathtt{CF}(\mathcal{C}) = \frac{1}{\sqrt{P_{av}(\mathcal{C})}} \cdot \min_{\mathbf{a}\in\mathcal{C}} \max_{t\in[0,1)} |F_{\mathbf{a}}(t)| \,. \tag{2.11}$$

The expressions become especially simple if all the signals have the same energy n or when the coefficients are drawn independently from a constellation $\mathcal{Q}$ scaled such that the average energy of a constellation point is 1. Then

$$\begin{aligned} \mathtt{PAPR}(\mathcal{C}) &= \frac{1}{n} \cdot \min_{\mathbf{a}\in\mathcal{C}} \mathtt{PAPR}(\mathbf{a}), \\ \mathtt{PMEPR}(\mathcal{C}) &= \frac{1}{n} \cdot \min_{\mathbf{a}\in\mathcal{C}} \mathtt{PMEPR}(\mathbf{a}), \\ \mathtt{CF}(\mathcal{C}) &= \frac{1}{\sqrt{n}} \cdot \min_{\mathbf{a}\in\mathcal{C}} \mathtt{CF}(\mathbf{a}). \end{aligned}$$

We measure PAPR and PMEPR in decibels (dB), using $10 \lg \mathtt{PAPR}(\mathbf{a})$ or $10 \lg \mathtt{PMEPR}(\mathbf{a})$ as an indication of the quality of a signal $\mathbf{a}$. Correspondingly, for CF we will replace the factor of 10 by 20. Notice that the maximum of an MC signal in an n-carrier system is n, thus the maximum of PAPR and PMEPR is n while the maximum CF is $\sqrt{n}$, which corresponds to $10 \lg n$.

Example 2.1 Let $n = 4$, and $\mathbf{a} = (1, -1, -1, 1)$. Then

$$F_{\mathbf{a}}(t) = 1 - e^{2\pi\imath t} - e^{4\pi\imath t} + e^{6\pi\imath t}.$$

The maximum of $|F_{\mathbf{a}}(t)|^2$ can be determined as follows. Indeed,

$$\begin{aligned} |F_{\mathbf{a}}(t)|^2 = {} & (1 - \cos 2\pi t - \cos 4\pi t + \cos 6\pi t)^2 \\ & + (-\sin 2\pi t - \sin 4\pi t + \sin 6\pi t)^2. \end{aligned}$$

Differentiating and equating the result to zero, after simple trigonometric manipulations, we reduce the problem to finding a solution to

$$1 + 2\cos 2\pi t - 3\cos^2 2\pi t = 0.$$

Thus the maximum of $|F_{\mathbf{a}}(t)|^2$ occurs when

$$t = \frac{1}{2\pi} \arccos\left(-\frac{1}{3}\right) = 0.304086,$$

and equals 9.48148. This corresponds to

$$10 \lg \frac{9.48148}{4} = 3.74816 \,\text{dB}.$$

□

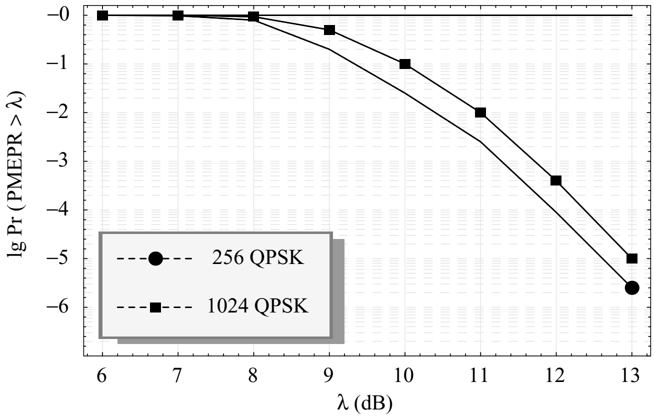

Figure 2.4 CCDFs of PMEPR of MC signal with 256 and 1024 QPSK modulated subcarriers

Another widely used characterization of MC signals deals with the probabilistic distribution of peak power values. Namely, we will be using the complementary cumulative distribution function (CCDF) of PAPR, PMEPR or CF. For instance, in the case of PMEPR, CCDF is just the probability that PMEPR of a randomly chosen MC signal exceeds a predefined threshold λ, $\Pr(\mathtt{PMEPR} > \lambda)$. An example of such CCDF for a QPSK modulated MC system with 256 and 1024 subcarriers is presented in Fig. 2.4.

Example 2.2 For $n = 4$ and BPSK modulation, Table 2.1 contains the list of PMEPR values for all possible sequences. In this situation (in contrast with large values of n) we are able to find the explicit distribution of probabilities of PMEPR,

PMEPR dB	2.48	3.75	6.02
Probability	1/2	1/4	1/4

□

There are several possible goals of the peak power control:

- Restriction of the set of used MC signals to those with peaks not exceeding a prescribed level;
- Restriction of the set of used MC signals in such a way that the probability of having a peak exceeding a prescribed level is much smaller than in the unrestricted set;
- Modification of the used MC signals in such a way that the probability of errors in the reconstructed coefficient vector is small, while at the same time the peaks are bounded with high probability.

Which of the goals is to be addressed depends on the system requirements and regulations.

Table 2.1 *PMEPRs for $n = 4$ and BPSK modulation*

Vector **a**				PMEPR(**a**) (dB)
−1	−1	−1	−1	6.02
1	−1	−1	−1	2.48
−1	1	−1	−1	2.48
1	1	−1	−1	3.75
−1	−1	1	−1	2.48
1	−1	1	−1	6.02
−1	1	1	−1	3.75
1	1	1	−1	2.48
−1	−1	−1	1	2.48
1	−1	−1	1	3.75
−1	1	−1	1	6.02
1	1	−1	1	2.48
−1	−1	1	1	3.75
1	−1	1	1	2.48
−1	1	1	1	2.48
1	1	1	1	6.02

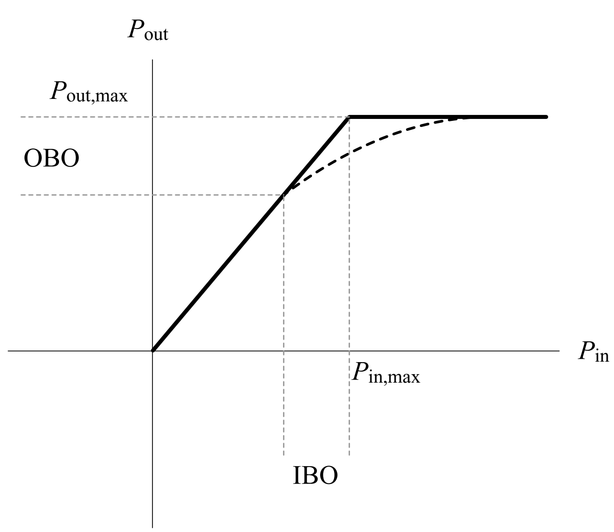

Figure 2.5 Input–output power characteristic of a HPA

2.3 Efficiency of power amplifiers

The perfectly linear ideal memoryless amplifier produces an output that is a multiple of the input. In reality, there is no amplifier able to provide unlimited output. The amplifier output is always limited to some value, called *saturation*. A typical characteristic of a HPA is presented in Fig. 2.5.