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168 The Cube: A Window to Convex and Discrete Geometry

THE CUBE: A WINDOW TO CONVEX AND DISCRETE GEOMETRY

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Preface

What is the simplest object in an n-dimensional Euclidean space? The answer should be a unit cube or a unit ball. On the one hand, compared with others, they can be easily described and intuitively imagined; on the other hand, they are perfect in shape with respect to symmetry and regularity. However, in fact, neither of them is really simple. For example, to determine the density of the densest ball packing is one of the most challenging problems in mathematics. As for the unit cube, though no open problem is as famous as Kepler's conjecture, there are many fascinating problems of no less importance.

What is the most important object in the n-dimensional Euclidean space? The answer should be the unit cube and the unit ball as well. Since, among other things, the unit ball is key to understanding metrics and surface area, and the unit cube is key to understanding measure and volume, in addition, a high-dimensional unit cube is rich in structure and geometry. Deep understanding of the unit cube is essential to understanding combinatorics and n-dimensional geometry.

This book has two main purposes: to show what is known about the cube and to demonstrate how analysis, combinatorics, hyperbolic geometry, number theory, algebra, etc. can be applied to the study of the cube.

The first two chapters discuss the area of a cross section and the area of a projection of the cube. The key problems in these directions are: *What is the maximal* (*minimal*) *area of a k-dimensional cross section of an n-dimensional unit cube? What is the maximal* (*minimal*) *area of a k-dimensional projection of an n-dimensional unit cube?* These problems are very natural and simple sounding. However, their solutions have not been completely discovered yet! On the other hand, it will be shown how deep analysis and linear algebra can deal with problems of these types.

The next two chapters study the maximal simplex inscribed in the cube and the triangulation problem. For example: *What is the maximum volume* *of a k-dimensional simplex that can be inscribed into an n-dimensional unit cube? What is the smallest number of simplices needed to triangulate an n-dimensional cube?* These problems are very combinatorial in nature. The inscribed simplex problem in fact is closely related to Hadamard matrices. However, to have a deep understanding of these problems, we have to apply knowledge of number theory and even hyperbolic geometry! In addition, so far our knowledge about these problems is very limited.

Then we discuss the 0/1 polytopes. Clearly 0/1 polytopes do provide nice models for finite geometry, geometric combinatorics, and optimization. However, what we want to emphasize here is the connection with coding theory. In fact, many problems about binary codes are problems about 0/1 polytopes!

The last three chapters deal with Minkowski's conjecture, Furtwängler's conjecture, and Keller's conjecture, respectively. The conjectures themselves are fascinating; moreover their algebraic solutions are just magic! According to S. K. Stein, "which in total are almost as startling as the metamorphosis of a caterpillar to a butterfly." Is there any other fascinating mathematical problem solved in a more surprising way? I know not one.

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Basic notation

E^n	Euclidean space of n-dimensions.
$\mathbf{x}$	A point (or a vector) of E^n with coordinates $(x_1, x_2, \ldots, x_n)$; or an element of a group.
$\mathbf{o}$	The origin of E^n.
$\langle \mathbf{x}, \mathbf{y} \rangle$	The inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$.
$\Vert \mathbf{x}, \mathbf{y} \Vert$	The Euclidean distance between two points $\mathbf{x}$ and $\mathbf{y}$.
$\Vert \mathbf{x} \Vert$	The Euclidean norm of $\mathbf{x}$.
X	A set of points in E^n.
$\mathrm{conv}\{X\}$	The convex hull of X.
$\mathrm{int}(X)$	The interior of X.
$v(X)$	The volume of X.
$s(X)$	The surface area of X.
$v_k(X)$	The k-dimensional measure of X.
K	An n-dimensional convex body.
$W_i(K)$	The ith quermassintegral of K.
I^n	The n-dimensional unit cube $\left\{\mathbf{x} : \ |x_i| \leq \frac{1}{2}\right\}$.
$\overline{I^n}$	The n-dimensional unit cube $\{\mathbf{x} : \ 0 \leq x_i \leq 1\}$.
B^n	The n-dimensional unit ball centered at $\mathbf{o}$.
ω_n	The volume of an n-dimensional unit ball.
P	An n-dimensional polytope.
$V(P)$	The set of the vertices of P.
Z	The set of all integers.
R	The set of all real numbers.
Λ	An n-dimensional lattice.
Z^n	The n-dimensional lattice $\{(z_1, z_2, \ldots, z_n) : z_i \in Z\}$.
$\alpha(n, k)$	The maximal volume of a k-dimensional cross section of I^n.
$\beta(n, k)$	The maximal volume of a k-dimensional projection of I^n.

$\theta(n, k)$	The maximal value of $\det(AA')$, where A is a $k \times n$ binary matrix. Especially, we abbreviate $\theta(n, n)$ to θ_n.
$\theta^*(n, k)$	The maximal value of $\det(AA')$, where A is a $k \times n$ matrix with ± 1 elements. Especially, we abbreviate $\theta^*(n, n)$ to θ_n^*.
$\gamma(n, k)$	The maximal volume of a k-dimensional simplex inscribed in I^n. Especially, we abbreviate $\gamma(n, n)$ to γ_n.
τ_n	The minimal cardinality of a triangulation of I^n.
$A(n, s)$	The maximal cardinality of an n-dimensional binary code with separation s.
$\phi(n)$	The number of the n-dimensional 0/1 polytopes reduced from $\overline{I^n}$.
$\langle \mathbf{g} \rangle$	The cyclic group generated by $\mathbf{g}$.
$\lvert \mathbf{g} \rvert$	The order of $\mathbf{g}$.
Z_k	The additive cyclic group $\{0, 1, \dots, k-1\}$ modulo k.
O_k	The multiplicative cyclic group $\left\{1, e^{2\pi i/k}, \dots, e^{2(k-1)\pi i/k}\right\}$, where $i = \sqrt{-1}$.
$\Re(G)$	The group ring generated by a group G.