Introduction

Let E^n denote the n-dimensional *Euclidean space* over the real number field R and with an *orthonormal basis* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Then each point $\mathbf{x}$ of E^n can be uniquely expressed as

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n = (x_1, x_2, \ldots, x_n),$$

where x_i is known as the ith *coordinate* of $\mathbf{x}$ with respect to the basis. Let $\langle \mathbf{u}, \mathbf{v} \rangle$ denote the *inner product* of two vectors $\mathbf{u}$ and $\mathbf{v}$ and let $\|\mathbf{x}, \mathbf{y}\|$ denote the *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$, by which the *Euclidean metric* is defined. With the coordinates of the vectors and the points, we can write

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i$$

and

$$\|\mathbf{x}, \mathbf{y}\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

For convenience, we abbreviate $\|\mathbf{x}, \mathbf{o}\|$ to $\|\mathbf{x}\|$. Let θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$, then we have

$$\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \cos\theta.$$

Clearly two vectors are orthogonal to each other if and only if their inner product is zero.

Now let us introduce two particular objects in E^n, namely

$$I^n = \left\{ \mathbf{x} \in E^n : \ |x_i| \leq \tfrac{1}{2} \text{ for all } i \right\}$$

and

$$B^n = \left\{ \mathbf{x} \in E^n : \ \|\mathbf{x}\| \leq 1 \right\}.$$

A subset of E^n is called an *n-dimensional unit cube* if it is congruent to I^n, and is called an *n-dimensional unit ball* if it is congruent to B^n. For convenience of the forthcoming usage, we introduce another particular unit cube

$$\overline{I^n} = \left\{ \mathbf{x} \in E^n : \ 0 \le x_i \le 1 \text{ for all } i \right\}.$$

In fact, $\overline{I^n}$ is a translate of I^n with a translative vector $\mathbf{v} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

In the n-dimensional Euclidean space, the *volume* $v(X)$ of a set X is its *Lebesgue measure*; that is

$$v(X) = \int_{E^n} \chi(\mathbf{x})\, d\mathbf{x},$$

where $\chi(\mathbf{x})$ is the *characteristic function* of the set. For the unit cube I^n and the unit ball B^n, we have

$$v(I^n) = 1 \tag{0.1}$$

and

$$v(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}, \tag{0.2}$$

where $\Gamma(x)$ is the *gamma function*. For convenience, we will abbreviate the volume of the n-dimensional unit ball to ω_n. In fact, (0.1) is the foundation in defining the measure of a general set in E^n.

For two subsets C and D of E^n, we define their *Minkowski sum* as

$$C + D = \left\{ \mathbf{x} + \mathbf{y} : \ \mathbf{x} \in C, \ \mathbf{y} \in D \right\}.$$

Then the *surface area* $s(X)$ of the set X is defined by

$$s(X) = \lim_{\epsilon \to 0} \frac{v(X + \epsilon B^n) - v(X)}{\epsilon},$$

if the limit does exist. Intuitively speaking, the set $X + \epsilon B^n$ is nothing else but the result of putting a tight coat of thickness ϵ on X. Applying this formula to I^n and B^n, respectively, we can easily deduce

$$s(I^n) = 2n$$

and

$$s(B^n) = n \cdot \omega_n.$$

A subset K of E^n is *convex* if

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in K,$$

whenever both $\mathbf{x}$ and $\mathbf{y}$ belong to K and $0 < \lambda < 1$. In addition, if it is also compact, we call it a *convex body*. For example, all balls, cubes, and simplices are convex bodies. Let $\mathbf{x}$ and $\mathbf{y}$ be two points of the unit ball B^n and let λ be a number satisfying $0 < \lambda < 1$. Then, by the *Cauchy–Schwarz inequality* we get

$$\begin{aligned}\|\lambda\mathbf{x}+(1-\lambda)\mathbf{y}\| &= \left(\sum_{i=1}^{n}(\lambda x_i+(1-\lambda)y_i)^2\right)^{1/2}\\ &\le \lambda\left(\sum_{i=1}^{n}x_i^2\right)^{1/2}+(1-\lambda)\left(\sum_{i=1}^{n}y_i^2\right)^{1/2}\\ &\le 1.\end{aligned}$$

Therefore, the unit ball is indeed convex. The convexity of the cubes and the simplices can be deduced by similar routine arguments.

There is another important concept about convexity, which will be useful in this book; namely, the *convex hull* of a given set X, which is defined by

$$\mathrm{conv}\{X\} = \left\{\sum \lambda_i\mathbf{x}_i : \ \mathbf{x}_i \in X; \ \lambda_i \ge 0, \ \sum \lambda_i = 1\right\}.$$

In fact, by *Carathéodory's theorem*, we can restrict each of the sum over only $n+1$ terms. In particular, if $\mathrm{card}\{X\}$ is a finite number, then $\mathrm{conv}\{X\}$ is a *convex polytope*. For example, both a cube and a simplex are polytopes and they are the convex hulls of the sets of their vertices. Let H be a supporting hyperplane of an n-dimensional polytope P. We call $F = P \cap H$ a *k-face* of P if it is k-dimensional. In particular, an $(n-1)$-face is called a *facet* and a 0-face is a vertex.

Let Z denote the *ring of integers* and let $\mathbf{a}_1, \ \mathbf{a}_2, \ldots, \mathbf{a}_n$ be n linearly independent vectors in E^n, then the set

$$\Lambda = \left\{\sum_{i=1}^{n} z_i\mathbf{a}_i : \ z_i \in Z\right\}$$

is called an n-dimensional *lattice* and $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is called a *basis* of the lattice. It is easy to see that a lattice is a *free abelian group* under addition and there are many different bases for a given lattice. For example

$$Z^n = \left\{(z_1, z_2, \ldots, z_n) : \ z_i \in Z\right\}$$

is an n-dimensional lattice with a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. In addition, the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for Z^n if and only if (u_{ij}) is a unimodular integral matrix, where $\mathbf{u}_i = (u_{i1}, u_{i2}, \ldots, u_{in})$.

To end this brief introduction, let us have a close look at the unit cube. Clearly, an n-dimensional unit cube is a cylinder of height 1 over an $(n-1)$-dimensional one. In other words

$$I^n = I^1 \oplus I^{n-1}. \tag{0.3}$$

Therefore, it is easy to see that every k-face $(0 \le k \le n-1)$ of I^n is a k-dimensional unit cube. Let $f(n,k)$ denote the number of the k-faces of I^n. By (0.3) we get

$$f(n,k) = 2f(n-1,k) + f(n-1,k-1). \tag{0.4}$$

Then, by induction on n and the identity

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k},$$

it can be deduced from (0.4) that

$$f(n,k) = 2^{n-k}\binom{n}{k}.$$

As a conclusion, when $0 \le k \le n-1$, the unit cube I^n has exactly $2^{n-k}\binom{n}{k}$ k-faces, each of which is a k-dimensional unit cube.

1

Cross sections

1.1 Introduction

Let us start with the two-dimensional case. Let H^1 denote a straight line in E^2 passing through the origin $\mathbf{o}$ (a one-dimensional subspace of E^2) and let $\ell(I^2\bigcap H^1)$ denote the length of $I^2\bigcap H^1$. If, without loss of generality, H^1 intersects the boundary of I^2 at $(\frac{1}{2}, y)$ and $(-\frac{1}{2}, -y)$, then $|y| \le \frac{1}{2}$ and

$$\ell(I^2\bigcap H^1) = \sqrt{1+4y^2}.$$

Therefore, for every H^1, we have

$$1 \le \ell(I^2\bigcap H^1) \le \sqrt{2}, \tag{1.1}$$

where the lower bound is attained if and only if H^1 is an axis of E^2 and the upper bound is attained if and only if H^1 contains a pair of antipodal vertices of I^2.

In fact, the length of any segment contained in I^2 is at most $\sqrt{2}$. This statement can be deduced by the following argument. Let L be a segment contained in I^2. Since I^2 is centrally symmetric, the segment L', which is symmetric to L, is also contained in I^2. Then by convexity we get

$$\tfrac{1}{2}(L+L') \subset I^2.$$

Since L is parallel with L' and $\frac{1}{2}(L+L')$ contains the origin $\mathbf{o}$, by (1.1) we get

$$\ell(L) = \ell(\tfrac{1}{2}(L+L')) \le \sqrt{2}.$$

The three-dimensional case is more complicated and more interesting. First, we study the one-dimensional sections. Without loss of generality, we assume that H^1 intersects the boundary of I^3 at $(\frac{1}{2}, y, z)$ and $(-\frac{1}{2}, -y, -z)$. Then we have $|y| \le \frac{1}{2}$, $|z| \le \frac{1}{2}$, and

$$\ell(I^3\bigcap H^1) = \sqrt{1+4y^2+4z^2}.$$

Therefore, for every H^1, we get

$$1 \le \ell(I^3 \cap H^1) \le \sqrt{3}, \tag{1.2}$$

where the lower bound is attained if and only if H^1 is an axis of E^3 and the upper bound is attained if and only if H^1 contains a pair of antipodal vertices of I^3. As with the two-dimensional case, by symmetry and convexity we can deduce that the length of any segment contained in I^3 is at most $\sqrt{3}$.

Next, let us discuss the two-dimensional cross sections of I^3. Let $\mathbf{u}$ be a point on the boundary of I^3 and let H^2 denote the two-dimensional subspace $\{\mathbf{x}: \ \mathbf{x} \in E^3, \ \langle \mathbf{x}, \mathbf{u} \rangle = 0\}$. Since I^3 has six facets and every edge of $I^3 \cap H^2$ is an intersection of H^2 with one of the facets, by symmetry it follows that $I^3 \cap H^2$ is either a parallelogram or a hexagon.

Let $v_2(I^3 \cap H^2)$ denote the area of $I^3 \cap H^2$ and write

$$U_1 = \left\{\mathbf{u}: \ u_3 = \tfrac{1}{2}, \ u_1 \ge 0, \ u_2 \ge 0, \ u_1 + u_2 \le \tfrac{1}{2}\right\},$$

$$U_2 = \left\{\mathbf{u}: \ u_3 = \tfrac{1}{2}, \ u_1 \le \tfrac{1}{2}, \ u_2 \le \tfrac{1}{2}, \ u_1 + u_2 > \tfrac{1}{2}\right\},$$

and, for $i = 1, 2, 3$

$$F_i = \left\{\mathbf{x} \in E^3: \ x_i = \tfrac{1}{2}, \ |x_j| \le \tfrac{1}{2} \text{ for } j \ne i\right\}.$$

Now, we estimate $v_2(I^3 \cap H^2)$ by considering two cases.

Case 1. $\mathbf{u} \in U_1$. Then the corresponding plane H^2 does not intersect the relative interior of F_3 and therefore $I^3 \cap H^2$ is a parallelogram (see Figure 1.1). By projecting $I^3 \cap H^2$ on to F_3 we get

$$v_2(I^3 \cap H^2) = \frac{\sqrt{0.5^2 + u_1^2 + u_2^2}}{0.5} v_2(F_3).$$

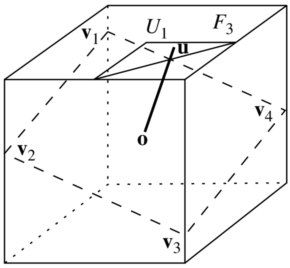

Figure 1.1

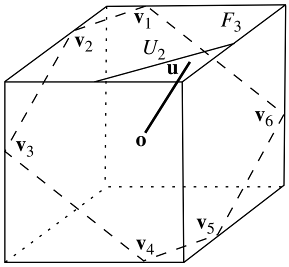

Figure 1.2

Thus, by a routine argument we can deduce

$$1 \leq v_2(I^3 \bigcap H^2) \leq \sqrt{2},$$

where the lower bound is attained if and only if $\mathbf{u} = (0, 0, \frac{1}{2})$ and the upper bound is attained if and only if $\mathbf{u} = (\frac{1}{2}, 0, \frac{1}{2})$ or $\mathbf{u} = (0, \frac{1}{2}, \frac{1}{2})$.

Case 2. $\mathbf{u} \in U_2$. Then the corresponding plane H^2 does intersect the relative interior of every facet of I^3 and therefore $I^3 \bigcap H^2$ is a hexagon (see Figure 1.2). Assume that H^2 does intersect the boundary of F_3 at two points $\mathbf{v}_1$ and $\mathbf{v}_2$. By a routine computation we can determine that $\mathbf{v}_1 = (\frac{2u_2-1}{4u_1}, -\frac{1}{2}, \frac{1}{2})$ and $\mathbf{v}_2 = (-\frac{1}{2}, \frac{2u_1-1}{4u_2}, \frac{1}{2})$. Since the projection of $I^3 \bigcap H^2$ to F_3 has area $1-(\frac{1}{2}+\frac{2u_2-1}{4u_1})(\frac{1}{2}+\frac{2u_1-1}{4u_2})$, we get

$$v_2(I^3 \bigcap H^2) = \sqrt{1+(2u_1)^2+(2u_2)^2}\left[1-\left(\frac{1}{2}+\frac{2u_2-1}{4u_1}\right)\left(\frac{1}{2}+\frac{2u_1-1}{4u_2}\right)\right].$$

For any fixed number c, it is easy to see that

$$1-\left(\frac{1}{2}+\frac{2u_2-1}{4u_1}\right)\left(\frac{1}{2}+\frac{2u_1-1}{4u_2}\right) = c$$

is a quadratic curve which is symmetric with respect to the straight line $u_1 = u_2$. Therefore, in this case $v_2(I^3 \bigcap H^2)$ attains its minimum and maximum on the boundary of U_2 or on the line $u_1 = u_2$. By checking these subcases we get

$$\sqrt{\frac{3}{2}} < v_2(I^3 \bigcap H^2) < \sqrt{2}.$$

As a conclusion, by symmetry, for every H^2 we have

$$1 \leq v_2(I^3 \bigcap H^2) \leq \sqrt{2}, \tag{1.3}$$

where the lower bound can be attained if and only if $\mathbf{u}$ is in the direction of an axis and the upper bound can be attained if and only if H^2 contains two pairs of antipodal vertices of I^3.

In fact, the area of any planar section of I^3 is at most $\sqrt{2}$. Let P be such a section. Then the set P', which is symmetric to P with respect to $\mathbf{o}$, is also a planar section of I^3. In addition, P' is parallel with P. Therefore, we have

$$\mathbf{o} \in \tfrac{1}{2}(P+P') \subset I^3$$

and, by the *Brunn–Minkowski inequality* and (1.3)

$$v_2(P) \leq v_2(\tfrac{1}{2}(P+P')) \leq \sqrt{2}.$$

These examples are relatively simple, at least they are manageable by elementary methods. However, similar problems in higher dimensions are much more challenging and fascinating. Let H^k denote a k-dimensional linear subspace of E^n containing $\mathbf{o}$ and let $v_k(S)$ denote the k-dimensional volume (measure) of a set S. The purpose of this chapter is to study the measure and the structure of $I^n \bigcap H^k$.

1.2 Good's conjecture

Based on the examples listed in the previous section, according to Hensley (1979), Anton Good made the following conjecture about the lower bound for $v_k(I^n \bigcap H^k)$.

Good's conjecture. *Let k be an integer satisfying $1 \leq k \leq n-1$. For every k-dimensional subspace H^k of E^n, we have*

$$v_k(I^n \bigcap H^k) \geq 1.$$

This conjecture is simple-sounding in nature. However, we have to use complicated analytic machinery to prove it. Let $\overline{B^k}$ denote the k-dimensional ball with radius

$$r = \frac{\Gamma(\frac{k}{2}+1)^{\frac{1}{k}}}{\pi^{\frac{1}{2}}}$$

and centered at the origin of E^k. Then by (0.2) we have

$$v_k(\overline{B^k}) = \omega_k \cdot r^k = \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} \cdot \frac{\Gamma(\frac{k}{2}+1)}{\pi^{\frac{k}{2}}} = 1.$$

Let $\chi(S, \mathbf{x})$ denote the characteristic function of a subset S of E^n. In 1979 J. D. Vaaler proved the following theorem.

Theorem 1.1 (Vaaler, 1979). *Suppose that $n_1, n_2, \ldots, n_j$ are positive integers, $n = n_1 + n_2 + \cdots + n_j$, $D = \overline{B^{n_1}} \oplus \overline{B^{n_2}} \oplus \cdots \oplus \overline{B^{n_j}} \subset E^n$, and A is a real $k \times n$ matrix with rank k. Then we have*

$$\int_{E^k} \chi(D, \mathbf{x}A)\, d\mathbf{x} \geq |AA'|^{-\frac{1}{2}}, \tag{1.4}$$

where A' is the transpose of A.

As usual, $X \oplus Y$ means the *Cartesian product* of X and Y. Let us take $n_1 = n_2 = \cdots = n_j = 1$ (then $j = n$) and choose A in this theorem so that its rows form an orthonormal basis for H^k in E^n. Then D is nothing else but the unit cube I^n and

$$\int_{E^k} \chi(I^n, \mathbf{x}A)\, d\mathbf{x} = \int_{H^k} \chi(I^n, \mathbf{y})\, d\mathbf{y} = v_k(I^n \bigcap H^k).$$

On the other hand, by the assumption, we get

$$|AA'| = 1.$$

Therefore, Good's conjecture follows as a corollary of Theorem 1.1.

Corollary 1.1 (Vaaler, 1979). *For every k-dimensional subspace H^k of E^n, we have*

$$v_k(I^n \bigcap H^k) \geq 1.$$

For a deep generalization of this result we refer to Meyer and Pajor (1988). Vaaler's theorem is very geometric. However, as one can imagine, its proof is very analytical. To introduce the proof, let us start with a couple of definitions.

Definition 1.1. *Let $f(\mathbf{x})$ be a nonnegative function defined in E^n. If*

$$f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \geq f(\mathbf{x}_1)^{\lambda} f(\mathbf{x}_2)^{1-\lambda}$$

holds for every pair of points $\mathbf{x}_1$ and $\mathbf{x}_2$ in E^n and every λ satisfying $0 < \lambda < 1$, then $f(\mathbf{x})$ is said to be logconcave.

Let $g(\mathbf{x})$ be a *concave function* defined in E^n and define $f(\mathbf{x}) = e^{g(\mathbf{x})}$. Since

$$g(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \geq \lambda g(\mathbf{x}_1) + (1-\lambda) g(\mathbf{x}_2)$$

holds for every pair of points $\mathbf{x}_1$ and $\mathbf{x}_2$ in E^n and every λ with $0 < \lambda < 1$, we have

$$\begin{aligned} f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) &= e^{g(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2)} \\ &\geq e^{\lambda g(\mathbf{x}_1)} \cdot e^{(1-\lambda) g(\mathbf{x}_2)} \\ &= f(\mathbf{x}_1)^{\lambda} f(\mathbf{x}_2)^{1-\lambda}. \end{aligned}$$

Thus $f(\mathbf{x})$ is logconcave in E^n.

Definition 1.2. *Let μ be a probability measure on E^n. If*

$$\mu(\lambda K_1 + (1-\lambda)K_2) \geq \mu(K_1)^{\lambda}\mu(K_2)^{1-\lambda}$$

holds for every pair of open convex sets K_1 and K_2 in E^n and every λ with $0 < \lambda < 1$, then μ is said to be logconcave in E^n.

As usual, we define the *support* of a probability measure μ to be the smallest closed subset S of E^n such that $\mu(E^n \setminus S) = 0$. In other words, the support of μ is the set of all points $\mathbf{x} \in E^n$ such that

$$\mu(rB^n + \mathbf{x}) > 0$$

for all $r > 0$. For convenience, it is denoted by $\mathrm{supp}(\mu)$.

Comparing with the logconcave functions, it seems more difficult to get examples for logconcave probability measures. In fact, as we can see from the following lemma, they are closely related.

Lemma 1.1 (Borell, 1975 and Prékopa, 1973). *Let μ be a logconcave probability measure on E^n and suppose that* $\mathrm{supp}(\mu)$ *spans a k-dimensional subspace H^k of E^n. Then there is a logconcave probability density function $f(\mathbf{x})$ defined on H^k such that $d\mu = f dv_k$, where v_k is the k-dimensional Lebesgue measure on H^k. On the other hand, for any logconcave probability density function $f(\mathbf{x})$ defined on a k-dimensional subspace H^k in E^n, the probability measure defined by $d\mu = f dv_k$ is logconcave.*

The first part of this lemma was proved by C. Borell, and the second part was proved by A. Prékopa. Both this result and the next lemma are intuitively imaginable. However, like many results in measure theory, their proofs are very complicated.

Proof (A sketch). Let $\mathbf{x}$ and $\mathbf{y}$ be two distinct points in H^k and let λ be a number satisfying $0 < \lambda < 1$. Since μ is a logconcave probability measure with a density function $f(\mathbf{x})$

$$\mu\left(\epsilon B^k + \lambda\mathbf{x} + (1-\lambda)\mathbf{y}\right) \geq \mu\left(\epsilon B^k + \mathbf{x}\right)^{\lambda} \mu\left(\epsilon B^k + \mathbf{y}\right)^{1-\lambda}$$

holds for all sufficiently small ϵ. Therefore, we can deduce

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq f(\mathbf{x})^{\lambda} f(\mathbf{y})^{1-\lambda},$$

which means that $f(\mathbf{x})$ is logconcave.

To prove the second part, we start with a basic inequality (see Prékopa, 1971 for the details). *If $g_1(x)$, $g_2(x)$, and $g(x)$ are nonnegative Borel measurable functions satisfying*

$$g(z) = \sup_{x+y=2z} g_1(x)\, g_2(y),$$