Introduction to Radon Transforms

The Radon transform represents a function on a manifold by its integrals over certain submanifolds. Integral transformations of this kind have a wide range of applications in modern analysis, integral and convex geometry, medical imaging, and many other areas. Reconstruction of functions from their Radon transforms requires tools from harmonic analysis and fractional differentiation. This comprehensive introduction contains a thorough exploration of Radon transforms and related operators when the basic manifolds are the real Euclidean space, the unit sphere, and the real hyperbolic space. Radon-like transforms are discussed not only on smooth functions but also in the general context of Lebesgue spaces. Applications, open problems, and recent results are also included.

The book will be useful for researchers in integral geometry, harmonic analysis, and related branches of mathematics, including applications. The text contains many examples and detailed proofs, making it accessible to graduate students and advanced undergraduates.

Boris Rubin is Professor of Mathematics at Louisiana State University. He is the author of the book *Fractional Integrals and Potentials*, and has written more than one hundred research papers in the areas of fractional calculus, integral geometry, and related harmonic analysis.

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With Elements of Fractional Calculus and Harmonic Analysis

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To my family

Contents

Preface

Perhaps some reader who has been charmed or fascinated by one or another exotic flower in the garden of offbeat integral geometry will be inspired to try his hand at cultivating a few blossoms of his own.

Lawrence Zalcman, Amer. Math. Monthly 87 (1980).

This book is intended as an introduction to integral transformations that take a function on some space X to a function on another space Y, which consists of geometric objects lying in X. The most important example is the Radon transform that assigns to a function on the Euclidean plane the integrals of this function over straight lines. Another example is the Funk transform that integrates functions on the two-dimensional sphere over great circles. Nowadays the term *Radon transform* has a wide meaning and is used for diverse transformations of that kind with different X and Y. Such transformations arise in integral and convex geometry, harmonic analysis, representation theory, and many other branches of mathematics. They form a mathematical background of modern tomography.

An idea of writing a friendly introductory book on Radon transforms and related operators of integral geometry evolved from numerous conversations with colleagues and students belonging to different mathematical schools. The selection of the material, the style of the book, and my research in the area were influenced by these meetings. My intention was to make the book accessible to different categories of the readers with standard background in analysis.

Inevitable time constraints and a huge variety of results in the area forced me to restrict the framework of the book. Thus a source space X has been chosen to be the Eucldean space $\mathbb{R}^n$, the unit sphere in $\mathbb{R}^n$, and the real hyperbolic space $\mathbb{H}^n$. The corresponding target space Y is restricted to codimension one objects. More general developments, related to totally geodesic submanifolds of arbitrary dimension, Grassmannians, matrix spaces, and others, are briefly mentioned in the notes to each chapter or can be found in references.

The distinctive feature of the book in comparison with other books on this subject is that it suggests to consider most of the Radon-like transforms as members of suitable analytic families of fractional integrals, which are of independent interest. This approach invokes diverse tools of fractional calculus and harmonic analysis for treating particular integral-geometrical problems. Furthermore, the Radon-like transforms are studied not only on smooth functions but also in the general context of Lebesgue spaces.

The structure of the book is the following. Chapter 1 includes necessary preliminaries. Chapters 2 and 3 provide an organized account of basics of fractional integration in one and several variables.

Chapter 4 plays the central role in the book. It is devoted to the hyperplane Radon transform and related operators on $\mathbb{R}^n$. In addition to the basic properties of the Radon transform, this chapter contains a detailed description of the related pioneering works by Ph. Mader, F. John, and A. A. Khachaturov, which are both of the historical interest and instructive.

Chapters 5 and 6 deal with Radon-like transforms on the unit sphere in $\mathbb{R}^n$ and on the real hyperbolic space. Spherical convolutions in Chapter 5 are especially important in integral geometry, for instance, in the solution of the Busemann-Petty problem for convex bodies (Section 5.4) and the injectivity problem for the spherical cap transform (Section 5.5).

Chapter 7 is devoted to several important spherical mean operators arising in the study of inverse problems for the Euler-Poisson-Darboux equation. Problems of this kind play an important role in medical imaging.

Appendix A contains basics of harmonic analysis on the sphere. Some open problems are discussed in Appendix B.

There are thousands of publications on Radon-like transforms, and the bibliography at the end of the book is far from being exhaustive. The references have been limited, when possible, to suitable publications containing further information. More references can be found in the books by Deans [122], Ehrenpreis [143], Epstein [144], Gardner [188], Gelfand, Gindikin, and Graev [195], Gelfand, Graev, and Vilenkin [205], Groemer [251], Gonzalez [224], Helgason [271, 275, 276], Koldobsky [324], Kuchment [337], Markoe [397], Natterer [422], Palamodov [450], Ramm and Katsevich [489], Schneider [581], Schneider and Weil [582], Valery V. Volchkov [651], and Valery V. Volchkov and Vitaly V. Volchkov [652].

Different chapters of the book can serve as a basis of lecture courses.

Guidelines to the Reader

Familiarity with preliminaries in Chapter 1 is desirable. The results from Chapter 2 related to the existence and inversion of fractional integrals are used in Chapters 4–7. Subsections 4.12.1 and 4.12.2 rely on the material of Subsection 2.6.3. The properties of Riesz potentials and the Semyanistyi-Lizorkin spaces (Chapter 3) are needed in Chapter 4. Chapters 4–7 are almost independent. However, the results of Subsection 4.12.2 related to the kernel and support theorems for the Radon transform are used in Chapters 5–7 to obtain similar statements for the corresponding projectively equivalent transformations. Chapter 5 assumes familiarity with basics of analysis on the sphere, as in Appendix A.

Dedication and Acknowledgments

This book is dedicated to my family and to the memory of my father Semyon Efimovich Rubin, a man of great erudition, who stimulated my curiosity and interest in research.

I attempted to give credits for all results in the book and the historical developments of the subject. My apologies go to authors whose works have not been mentioned. Some chapters of the book have been used in my lecture courses. I am very grateful to students for their numerous questions, comments, and patience.

I am indebted to my first teacher, Stefan Grigorievich Samko, for introducing me into the realm of fractional calculus and related branches of analysis. My favorite professors, Igor Borisovich Simonenko, Viktor Iosifovich Judovich, and Vyacheslav Pavlovich Zakharyuta, encouraged me in my first steps as a researcher and taught me to see the beauty of Mathematics. Inspiring conversations with Sigurdur Helgason, his kindness, and advice stimulated my interest in Radon transforms and integral geometry.

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Diverse results included in the book were repeatedly discussed in the harmonic analysis seminar at the Department of Mathematics, Louisiana State University, and in the functional analysis seminar at the Einstein Institute of Mathematics, the Hebrew University of Jerusalem. I am grateful to all of the participants of these seminars for fruitful discussions and remarks. Thanks should be added to organizers and participants of many other seminars and conference sessions, where I had a privilege to give presentations.

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Notation and Conventions

$\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{C}$ – the sets of all integers, positive integers, real numbers, and complex numbers, respectively;

$[a]$ – the integer part of $a \in \mathbb{R}$;

$\{a\} = a - [a] \in [0, 1)$;

$\mathbb{Z}_+ = \{j \in \mathbb{Z} : \ j \geq 0\}$;

$\mathbb{R}_+ = \{a \in \mathbb{R} : \ a > 0\}$;

$\mathbb{R}^n = \{x = (x_1, \dots, x_n) : \ x_j \in \mathbb{R}, \ j = 1, \dots, n\}$; $e_1, \dots, e_n$ are the coordinate unit vectors in $\mathbb{R}^n$;

$x \cdot y = x_1 y_1 + \cdots + x_n y_n, \quad |x| = (x_1^2 + \cdots + x_n)^{1/2}$;

$\mathbb{Z}_+^n = \{\gamma = (\gamma_1, \dots, \gamma_n) : \ \gamma_j \in \mathbb{Z}_+ \ \forall j = 1, \dots, n\}$ – the set of multi-indices; $|\gamma| = \gamma_1 + \cdots + \gamma_n$ – the length of the multi-index γ;

$x^\gamma = x_1^{\gamma_1} \dots x_n^{\gamma_n}$;

$\partial_i = \partial/\partial x_i, \quad \partial^\gamma = \partial_1^{\gamma_1} \dots \partial_n^{\gamma_n}$;

$\Delta = \partial_1^2 + \cdots + \partial_n^2$ – the Laplace operator on $\mathbb{R}^n$;

$D = (2t)^{-1} d/dt = d/dt^2$;

$S^{n-1} = \{x \in \mathbb{R}^n : \ |x| = 1\}$ – the unit sphere in $\mathbb{R}^n$;

$\sigma_{n-1} = 2\pi^{n/2} \big/ \Gamma(n/2)$ – the surface area of S^{n-1};

$S_+^{n-1} = \{x \in S^{n-1} : \ x_n > 0\}$ – the "upper" hemisphere of S^{n-1};

$d\theta \ (\theta \in S^{n-1})$ – the surface element on S^{n-1};

$d_*\theta \ (= d\theta/\sigma_{n-1})$ – the normalized surface element on S^{n-1};

$\theta^\perp = \{x \in \mathbb{R}^n : \ x \cdot \theta = 0\}$ – the subspace of $\mathbb{R}^n$ orthogonal to $\theta \in S^{n-1}$;

$B_n = \{x \in \mathbb{R}^n : |x| < 1\}$ – the unit ball in $\mathbb{R}^n$;

$b_n = \sigma_{n-1}/n = \pi^{n/2}/\Gamma(n/2+1)$ – the volume of B_n;

$C_n = S^{n-1} \times [-1, 1]$ – the truncated cylinder in $\mathbb{R}^{n+1}$;

$\delta_{i,j}$ – the Kronecker delta; $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise;

$a.c.$ – analytic continuation.

Matrices

a^T – the transpose of the matrix a;

$M_{n,m}$ – the space of real matrices having n rows and m columns;

$\det(a)$ – the determinant of the matrix a;

I_n – the identity $n \times n$ matrix.

Groups

$GL(n)\,(\equiv GL(n;\mathbb{R}))$ – the general linear group of real $n \times n$ matrices γ satisfying $\det(\gamma) \neq 0$;

$O(n)$ – the group of orthogonal matrices $\gamma \in GL(n)$ satisfying $\gamma^T \gamma = I_n$;

$SO(n)$ – the special orthogonal group of matrices $\gamma \in O(n)$, for which $\det(\gamma) = 1$; $\dim SO(n) = \dim O(n) = n(n-1)/2$;

$M(n) = O(n) \times \mathbb{R}^n$ – the Euclidean motion group of $\mathbb{R}^n$ with group low $(\gamma, a) \cdot (\gamma', a') = (\gamma\gamma', \gamma a' + a)$, where $\gamma, \gamma' \in O(n)$ and $a, a' \in \mathbb{R}^n$. $M(n)$ can be identified with the group of matrices of the form $\begin{bmatrix} \gamma & a \\ 0 & 1 \end{bmatrix}$; $\dim M(n) = n(n+1)/2$.

Spaces

If Ω is an open set in $\mathbb{R}^n$, then

$C(\Omega)$ is the space of complex-valued continuous functions on Ω;

$C^m(\Omega) = \{f \in C(\Omega) : \partial^\gamma f \in C(\Omega) \ \ \forall |\gamma| \le m\}$;

$C^\infty(\Omega) = \bigcap_{m=1}^{\infty} C^m(\Omega)$;

$C_c^\infty(\Omega)$ is the space of C^∞ functions with compact support in Ω;

$C_0(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : \lim_{|x| \to \infty} f(x) = 0\}$;

$C_\mu(\mathbb{R}^n) = \{f : \ f \in C(\mathbb{R}^n), \ f(x) = O(|x|^{-\mu})\}, \quad \mu > 0$;

$C^m_\mu(\mathbb{R}^n) = \{f \in C^m(\mathbb{R}^n) : \ \partial^\gamma f \in C_\mu(\mathbb{R}^n) \ \forall \, |\gamma| \le m\}$;

$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \ x^\alpha \partial^\beta f(x) \in C_0(\mathbb{R}^n) \quad \forall \alpha, \beta \in \mathbb{Z}^n_+\}$ – the Schwartz space of test functions;

$S'(\mathbb{R}^n)$ – the space of tempered distributions;

$\Phi(\mathbb{R}^n) = \{f \in S(\mathbb{R}^n) : \ \int_{\mathbb{R}^n} x^\gamma f(x) dx = 0 \quad \forall \gamma \in \mathbb{Z}^n_+\}$ – the Semyanistyi-Lizorkin space;

$L^p(\Omega) = \left\{f : \ \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < \infty \right\}, \quad 1 \le p < \infty$;

$L^\infty(\Omega) = \left\{ f : \ \|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| < \infty \right\}$;

p' – the conjugate exponent to p : $1/p + 1/p' = 1$, $1 \le p \le \infty$ (if $p = \infty$, we set $1/p' = 0$);

$L^1_{loc}(\Omega)$ – the space of functions that are locally integrable on Ω, i.e., Lebesgue integrable on every compact subset of Ω;

$L^p(\Omega; w) = \{f : \ \|f\|_{L^p(\Omega; w)} = \|wf\|_{L^p(\Omega)} < \infty\}$;

$\mathcal{M}(X)$ – the space of finite complex Borel measures on the set X.

Selected Conventions

- The letter c (sometimes with subscripts and superscripts) denotes an inessential positive constant, which may vary at each occurrence.
- The script capital letter $\mathcal{W}$ (sometimes with subscripts) is used for diverse wavelet-like transforms, the precise meaning of which is clear from the context.
- All integrals are meant as Lebesgue integrals unless otherwise stated. We say that an integral under consideration exists in the Lebesgue sense if it is finite when the integrand is replaced by its absolute value.
- The following expressions are usually understood as Lebesgue integrals or in the sense of distributions, unless otherwise stated:

$$(f, \varphi) = \int_X f(x) \overline{\varphi(x)} \, dx, \qquad \langle f, \varphi \rangle = \int_X f(x) \varphi(x) \, dx.$$

- A function f on Ω is called sufficiently smooth if $f \in C^m(\Omega)$ for a sufficiently large m.
- The word "if" in definitions is always understood as "if and only if."