1

Preliminaries

The purpose of this introductory chapter is to present basic auxiliary facts from analysis that will be used throughout the book. For convenience of the reader, some statements are given with proofs. The corresponding references and related information can be found in Section 1.14.

1.1 Basic Integral Inequalities

- Hölder's inequality: For $1 \le p \le \infty$,

$$\left| \int_{\mathbb{R}^n} f_1(x) f_2(x)\, dx \right| \le \|f_1\|_p \, \|f_2\|_{p'}, \qquad 1/p + 1/p' = 1. \tag{1.1.1}$$

- The generalized Minkowski inequality: For $1 \le p \le \infty$,

$$\left\| \int_{\Omega_1} f(\cdot, y)\, dy \right\|_{L^p(\Omega_2)} \le \int_{\Omega_1} \|f(\cdot, y)\|_{L^p(\Omega_2)}\, dy. \tag{1.1.2}$$

- Young's inequality: Let

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$$

be the convolution of functions f and g on $\mathbb{R}^n$. Then

$$\|f * g\|_q \le \|f\|_p \|g\|_r, \tag{1.1.3}$$

$$1 \le p \le q \le \infty, \quad 1 - 1/p + 1/q = 1/r.$$

1.2 Integral Operators with Homogeneous Kernel

The following statements for operators with homogeneous kernel contain important ideas that will be extended in Chapter 4 to Radon-like transforms in integral geometry.

Theorem 1.1. *For x and y in $\mathbb{R}_+$, let $k(x, y)$ be a homogeneous function of degree -1, that is,*

$$k(\lambda x, \lambda y) = \lambda^{-1} k(x, y) \quad \forall \lambda > 0.$$

If at least one of the integrals

$$A_1 = \int\limits_{\mathbb{R}_+} |k(x, 1)|\, x^{-1/p'}\, dx, \qquad A_2 = \int\limits_{\mathbb{R}_+} |k(1, y)|\, y^{-1/p}\, dy, \tag{1.2.1}$$

is finite, then $A_1 = A_2\, (= A)$ and the operator $(Kf)(x) = \int_{\mathbb{R}_+} k(x, y) f(y)\, dy$ is bounded in $L^p(\mathbb{R}_+)$, $1 \le p \le \infty$, with $||K|| \le A$.

Proof. The coincidence of A_1 and A_2 is a consequence of the homogeneity of $k(x, y)$, which implies $k(x, 1) = x^{-1} k(1, x^{-1})$. Furthermore, changing variables and applying the generalized Minkowski inequality (1.1.2), we get

$$\begin{aligned} ||Kf||_p &= \Bigg|\Bigg| \int\limits_{\mathbb{R}_+} k(1, y) f(yx)\, dy \Bigg|\Bigg|_p \\ &\le \int\limits_{\mathbb{R}_+} |k(1, y)| \Bigg(\int\limits_{\mathbb{R}_+} |f(yx)|^p\, dx \Bigg)^{1/p} dy = A\, ||f||_p. \end{aligned}$$

□

Theorem 1.1 has the following generalization, which contains an explicit expression for the operator norm.

Theorem 1.2. *For x and y in $\mathbb{R}^n$, $n \ge 1$, let $k(x, y)$ be a homogeneous function of degree $-n$, that is,*

$$k(\lambda x, \lambda y) = \lambda^{-n} k(x, y) \quad \forall \lambda > 0. \tag{1.2.2}$$

Suppose also that k is invariant under orthogonal transformations:

$$k(\gamma x, \gamma y) = k(x, y) \quad \forall \gamma \in O(n). \tag{1.2.3}$$

If $e_1 = (1, 0, \dots, 0)$ and at least one of the integrals

$$B_1 = \int\limits_{\mathbb{R}^n} |k(x, e_1)|\, x^{-n/p'}\, dx, \qquad B_2 = \int\limits_{\mathbb{R}^n} |k(e_1, y)|\, y^{-n/p}\, dy,$$

is finite, then $B_1 = B_2\, (= B)$ and the operator $(Kf)(x) = \int_{\mathbb{R}^n} k(x, y) f(y)\, dy$ is bounded in $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, with $||K|| \le B$. If, moreover, $k(x, y)$ is nonnegative, then $||K|| = B$.

Proof. One can choose any other unit vector instead of e_1. To prove the coincidence of B_1 and B_2, we pass to polar coordinates and replace integration over S^{n-1} by integration over the group $O(n)$. Specifically, by (1.2.2) and (1.2.3),

$$\begin{aligned} B_2 &= \int_{\mathbb{R}_+} r^{n-1-n/p} dr \int_{S^{n-1}} |k(e_1, r\theta)| \, d\theta \\ &= \sigma_{n-1} \int_{\mathbb{R}_+} r^{n/p'-1-n} dr \int_{O(n)} |k(r^{-1}e_1, \gamma e_1)| \, d\gamma \\ &= \sigma_{n-1} \int_{\mathbb{R}_+} s^{n-1-n/p'} ds \int_{O(n)} |k(s\gamma^{-1}e_1, e_1)| \, d\gamma \\ &= \int_{\mathbb{R}_+} s^{n-1-n/p'} ds \int_{S^{n-1}} |k(s\theta, e_1)| \, d\theta = B_1. \end{aligned}$$

To estimate $(Kf)(x)$, let $x = r\gamma e_1$ where $r > 0$, $\gamma \in O(n)$, so that by (1.2.2) and (1.2.3),

$$(Kf)(r\gamma e_1) = r^{-n} \int_{\mathbb{R}^n} k(e_1, r^{-1}\gamma^{-1}y) f(y) \, dy = \int_{\mathbb{R}^n} k(e_1, z) f(r\gamma z) \, dz. \quad (1.2.4)$$

Then, by the generalized Minkowski inequality,

$$\begin{aligned} ||Kf||_p &= \left(\sigma_{n-1} \int_{\mathbb{R}_+} r^{n-1} dr \int_{O(n)} \left| \int_{\mathbb{R}^n} k(e_1, z) f(r\gamma z) \, dz \right|^p d\gamma \right)^{1/p} \\ &\le \int_{\mathbb{R}^n} |k(e_1, z)| \left(\sigma_{n-1} \int_{\mathbb{R}_+} r^{n-1} dr \int_{O(n)} |f(r\gamma z)|^p d\gamma \right)^{1/p} dz. \end{aligned}$$

Changing variable $r = |z|^{-1}s$, we get $||Kf||_p \le B \, ||f||_p$, which gives the first part of the theorem and also $||K|| \le B$.

It remains to show that $||K|| \ge B$ if $k(x, y)$ is nonnegative. Since K is a bounded operator in $L^p(\mathbb{R}^n)$, for any $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$ we obtain

$$\left| \int_{\mathbb{R}^n} (Kf)(x) g(x) \, dx \right| \le ||K|| \, ||f||_p \, ||g||_{p'}. \quad (1.2.5)$$

Assuming f and g radial and nonnegative and setting $f(x) = f_0(|x|)$, $g(x) = g_0(|x|)$, we write (1.2.5) as

$$\begin{aligned} &\int_{\mathbb{R}_+} r^{n-1} g_0(r) \, dr \int_{\mathbb{R}^n} k(e_1, z) f_0(r|z|) \, dz \\ &\le ||K|| \left(\int_{\mathbb{R}_+} r^{n-1} f_0^p(r) \, dr \right)^{1/p} \left(\int_{\mathbb{R}_+} r^{n-1} g_0^{p'}(r) \, dr \right)^{1/p'}. \end{aligned}$$

Then we choose

$$f_0(r) = \begin{cases} 0 & \text{if } r < 1, \\ r^{-(n+\varepsilon)/p} & \text{if } r > 1, \end{cases} \qquad g_0(r) = \begin{cases} 0 & \text{if } r < 1, \\ r^{-(n+\varepsilon)/p'} & \text{if } r > 1, \end{cases} \qquad \varepsilon > 0.$$

A simple calculation yields

$$\int_{\mathbb{R}^n} k(e_1, z)|z|^{-(n+\varepsilon)/p}\, dz \int_{\max(1,1/|z|)}^{\infty} r^{-1-\varepsilon}\, dr \le \frac{||K||}{\varepsilon},$$

or

$$\int_{|z|<1} k(e_1, z)\,|z|^{-n/p+\varepsilon/p'}\, dz + \int_{|z|>1} k(e_1, z)\,|z|^{-n/p-\varepsilon/p}\, dz \le ||K||.$$

Passing to the limit as $\varepsilon \to 0$, we obtain $B \le ||K||$. □

1.3 Analyticity of Functions Represented by Integrals

The following lemma is useful for proving analyticity of integrals of the form

$$\mathcal{F}(z) = \int_{\Omega} f(x, z)\, d\mu(x),$$

where Ω is a subset of a measure space possessing a nonnegative measure μ.

Lemma 1.3. *Let $f(x, z)$ be an analytic function of $z \in D \subseteq \mathbb{C}$ for μ-almost all $x \in \Omega$. If there is a function $g_0(x) \in L^1(\Omega, |\mu|)$ such that $|f(x, z)| \le g_0(x)$ for μ-almost all $x \in \Omega$ and all $z \in D$, then $\mathcal{F}(z)$ is an analytic function on D.*

Proof. Let C be a circle of radius R, belonging to D and centered at the point z, and let $l(C)$ be the length of C. Then

$$\mathcal{F}(z) = \frac{1}{2\pi i} \int_{\Omega} d\mu(x) \int_C \frac{f(x, t)}{t - z}\, dt \quad \text{and} \quad \frac{\partial f(x, z)}{\partial z} = \frac{1}{2\pi i} \int_C \frac{f(x, t)}{(t - z)^2}\, dt.$$

Denote

$$\mathcal{F}_1(z) = \frac{\mathcal{F}(z + \Delta z) - \mathcal{F}(z)}{\Delta z} - \frac{1}{2\pi i} \int_{\Omega} d\mu(x) \int_C \frac{f(x, t)}{(t - z)^2}\, dt, \qquad |\Delta z| < R.$$

Since

$$\begin{aligned} \frac{\mathcal{F}(z + \Delta z) - \mathcal{F}(z)}{\Delta z} &= \frac{1}{\Delta z} \int_{\Omega} \frac{d\mu(x)}{2\pi i} \int_C \left(\frac{f(x, t)}{t - (z + \Delta z)} - \frac{f(x, t)}{t - z} \right) dt \\ &= \frac{1}{2\pi i} \int_{\Omega} d\mu(x) \int_C \frac{f(x, t)}{(t - (z + \Delta z))(t - z)}\, dt, \end{aligned}$$

we obtain

$$|\mathcal{F}_1(z)| = \left| \frac{1}{2\pi i} \int_\Omega d\mu(x) \int_C \frac{f(x,t)\,\triangle z}{(t-z)^2(t-(z+\triangle z))}\, dt \right|$$

$$\leq \frac{|\triangle z|}{2\pi} \int_\Omega g_0(x)\, d|\mu|(x) \int_C \frac{|dt|}{|(t-z)^2(t-(z+\triangle z))|}$$

$$\leq \frac{|\triangle z|\, l(C)}{2\pi R^2(R-|\triangle z|)} \int_\Omega g_0(x)\, d|\mu|(x) \to 0 \quad \text{as } |\triangle z| \to 0.$$

It follows that

$$\mathcal{F}'(z) = \frac{1}{2\pi i} \int_\Omega d\mu(x) \int_C \frac{f(x,t)}{(t-z)^2}\, dt.$$

This integral converges because

$$\left| \int_C \frac{f(x,t)}{(t-z)^2}\, dt \right| \leq \frac{l(C)}{R^2}\, g_0(x) \in L^1(\Omega, |\mu|).$$

Thus $\mathcal{F}(z)$ is differentiable in D, and therefore it is analytic in D. □

1.4 Gamma and Beta Functions

For $Re\, z > 0$, the gamma function $\Gamma(z)$ is defined as an absolutely convergent improper integral

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx. \tag{1.4.1}$$

The equality

$$\Gamma(z+1) = z\Gamma(z) \tag{1.4.2}$$

extends $\Gamma(z)$ to the entire complex plane $\mathbb{C}$ as a meromorphic function with a simple pole at the points $z = 0, -1, -2, \ldots$. The reciprocal $1/\Gamma(z)$ is an entire function having zero of the first order at the same points.

The following formulas are known:

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z) \text{ (the reflection formula)}, \tag{1.4.3}$$

$$\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z)\Gamma(z+1/2) \text{ (the duplication formula)}, \tag{1.4.4}$$

$$\Gamma(n+1) = n! \text{ (n is a nonnegative integer)}. \tag{1.4.5}$$

For any complex numbers a and b and any integer $N > 0$, the asymptotic relation

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = \sum_{n=0}^{N-1} c_n(\alpha, b)\, z^{a-b-n} + O(z^{a-b-N}), \qquad z \to \infty, \tag{1.4.6}$$

holds provided that $z \notin \{-a, -a-1, -a-2, \dots\} \cup \{-b, -b-1, -b-2, \dots\}$. The coefficients $c_n(\alpha, b)$ can be explicitly computed. In particular,

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \tag{1.4.7}$$

under the same assumptions for z, as in (1.4.6).

The *Pochhammer symbol* $(\lambda)_k$ is defined by

$$(\lambda)_k = \lambda(\lambda+1)\dots(\lambda+k-1) = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}. \tag{1.4.8}$$

The beta function is defined as an integral

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx, \qquad Re\,\alpha > 0,\ Re\,\beta > 0, \tag{1.4.9}$$

and satisfies

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.4.10}$$

The binomial coefficients

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}; \qquad m, n \in \mathbb{Z}_+, \quad n \geq m,$$

admit the following generalization to arbitrary complex α and β provided that $\alpha \neq -1, -2, \dots$:

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)} = \frac{\sin(\beta-\alpha)\pi}{\pi}\,\frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha)}{\Gamma(\beta+1)}. \tag{1.4.11}$$

In particular, for $\beta = m \in \mathbb{Z}_+$,

$$\binom{\alpha}{m} = \frac{(-1)^{m-1}\alpha\Gamma(m-\alpha)}{\Gamma(1-\alpha)\Gamma(m+1)} = \frac{\sin(m-\alpha)\pi}{\pi}\,\frac{\Gamma(\alpha+1)\Gamma(m-\alpha)}{\Gamma(m+1)}. \tag{1.4.12}$$

By (1.4.7), for $\alpha \notin \mathbb{Z}_+$ we have

$$\binom{\alpha}{m} = O(m^{-\alpha-1}), \qquad m \to \infty. \tag{1.4.13}$$

1.5 Finite Differences

Let φ be a function on $\mathbb{R}^n$. The expression

$$(\Delta_y^\ell \varphi)(x) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \varphi(x-ky) \tag{1.5.1}$$

is called a *finite difference* of φ of order ℓ with step y at the point x. One can readily see that

$$(\Delta_y^1 \varphi)(x) = \varphi(x) - \varphi(x-y), \qquad (\Delta_y^\ell \varphi)(x) = (\Delta_y^1 \Delta_y^{\ell-1} \varphi)(x),$$

or

$$\Delta_y^\ell = (I - \tau_y)^\ell,$$

where I is the identity operator and τ_y stands for the translation $(\tau_y\varphi)(x) = \varphi(x - y)$.

Lemma 1.4. *If $\varphi \in C^l(\mathbb{R}^n)$, $y = t\theta, t > 0, \theta \in S^{n-1}$, then*

$$(\Delta_y^\ell \varphi)(x) = (-1)^\ell \int_0^{|y|} dt_\ell \int_0^{|y|} dt_{\ell-1} \ldots \int_0^{|y|} (\partial_t^\ell \varphi)(x - t\theta)\Big|_{t=t_1+\ldots+t_\ell} dt_1. \qquad (1.5.2)$$

Proof. We have

$$(\Delta_y^1 \varphi)(x) = \varphi(x) - \varphi(x-y) = -\int_0^{|y|} (\partial_{t_1}\varphi)(x - t_1\theta)\, dt_1,$$

$$(\Delta_y^2 \varphi)(x) = -\Delta_y^1 \left[\int_0^{|y|} (\partial_{t_1}\varphi)(x - t_1\theta)\, dt_1 \right]$$

$$= (-1)^2 \int_0^{|y|} dt_2 \int_0^{|y|} (\partial_t^2 \varphi)(x - t\theta)\Big|_{t=t_1+t_2} dt_1,$$

and so on. □

1.6 Jacobi, Gegenbauer, and Chebyshev Polynomials

The Jacobi polynomials $P_m^{(\alpha,\beta)}(t)$, $m = 0, 1, 2, \ldots$, constitute an orthogonal system in the space $L^2([-1, 1]; w_{\alpha,\beta})$ with the weight

$$w_{\alpha,\beta}(t) = (1-t)^\alpha (1+t)^\beta, \qquad \alpha, \beta > -1.$$

They are standardly normalized by

$$P_m^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha + 1 + m)}{m!\,\Gamma(\alpha + 1)} \qquad (1.6.1)$$

and satisfy

$$\int_{-1}^{1} P_k^{(\alpha,\beta)}(t)\, P_m^{(\alpha,\beta)}(t)\, w_{\alpha,\beta}(t)\, dt = \delta_{k,m} h_k^{(\alpha,\beta)} \qquad (1.6.2)$$

with

$$h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2k + \alpha + \beta + 1} \frac{\Gamma(k + \alpha + 1)\,\Gamma(k + \beta + 1)}{k!\,\Gamma(k + \alpha + \beta + 1)}. \qquad (1.6.3)$$

Moreover,

$$P_m^{(\alpha,\beta)}(-t) = (-1)^k \, P_m^{(\beta,\alpha)}(t). \tag{1.6.4}$$

The Gegenbauer polynomials $C_m^\lambda(t)$ form an orthogonal system in the space $L^2([-1,1];w_\lambda)$, where $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$, $\lambda > -1/2$. In many occurrences it is convenient to treat the cases $\lambda \neq 0$ and $\lambda = 0$ separately.

For all $\lambda > -1/2$ with $\lambda \neq 0$, the following formulas hold.

$$C_m^\lambda(t) = \frac{\Gamma(2\lambda+m)\,\Gamma(\lambda+1/2)}{\Gamma(2\lambda)\,\Gamma(\lambda+1/2+m)}\, P_m^{(\lambda-1/2,\lambda-1/2)}(t) \tag{1.6.5}$$

$$= \frac{\Gamma(2\lambda+m)}{m!\,\Gamma(2\lambda)}\, F\left(-m, m+2\lambda; \lambda+\frac{1}{2}; \frac{1-t}{2}\right) \tag{1.6.6}$$

$$= \sum_{q=0}^{[m/2]} c_q\, t^{m-2q}, \qquad c_q = \frac{(-1)^q\, 2^{m-2q}\, \Gamma(\lambda+m-q)}{q!\,(m-2q)!\,\Gamma(\lambda)}, \tag{1.6.7}$$

where $F(a,b;c;z)$ is the Gauss hypergeometric function;

$$\int_{-1}^{1} C_\ell^\lambda(t)\, C_m^\lambda(t)\, w_\lambda(t)\, dt = \delta_{\ell,m} h_\ell^\lambda, \qquad h_\ell^\lambda = \frac{2^{1-2\lambda}\,\pi\,\Gamma(\ell+2\lambda)}{\ell!\,(\ell+\lambda)\,\Gamma(\lambda)^2}. \tag{1.6.8}$$

The corresponding Rodrigues formula has the form

$$C_m^\lambda(t) = c\,(1-t^2)^{1/2-\lambda} \left(\frac{d}{dt}\right)^m (1-t^2)^{\lambda+m-1/2}, \tag{1.6.9}$$

$$c = \frac{(-1)^m}{2^m\, m!}\, \frac{\Gamma(2\lambda+m)\,\Gamma(\lambda+1/2)}{\Gamma(2\lambda)\,\Gamma(\lambda+m+1/2)}.$$

These formulas imply the following properties.

$$C_m^\lambda(-t) = (-1)^m\, C_m^\lambda(t), \qquad C_0^\lambda(t) = 1, \qquad C_1^\lambda(t) = 2\lambda t; \tag{1.6.10}$$

$$C_m^\lambda(1) = \frac{(2\lambda)_m}{m!} = \frac{\Gamma(2\lambda+m)}{m!\,\Gamma(2\lambda)}; \tag{1.6.11}$$

$$C_m^\lambda(0) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ (-1)^{m/2}\, \dfrac{\Gamma(\lambda+m/2)}{\Gamma(\lambda)\,(m/2)!} & \text{if } m \text{ is even.} \end{cases} \tag{1.6.12}$$

Furthermore, for $|t| \le 1$,

$$|C_m^\lambda(t)| \le c \begin{cases} 1 & \text{if } m \text{ is even,} \\ |t| & \text{if } m \text{ is odd,} \end{cases} \qquad c \equiv c(\lambda, m) = \text{const}; \tag{1.6.13}$$

cf. 10.9(18) in Erdélyi [145].

In the case $\lambda = 0$, the Gegenbauer polynomials are usually substituted by the Chebyshev polynomials

$$T_m(t) = \frac{m}{2} \lim_{\lambda\to 0} \frac{1}{\lambda}\, C_m^\lambda(t) = F\left(-m, m; \frac{1}{2}; \frac{1-t}{2}\right) = \cos(m \arccos t), \tag{1.6.14}$$

which form an orthogonal system in $L^2([-1,1]; w_0)$, $w_0(t) = (1-t^2)^{-1/2}$. Specifically,

$$\int_{-1}^{1} T_\ell(t)\, T_m(t)\, w_0(t)\, dt = \delta_{\ell,m} \begin{cases} \pi/2 & \text{if } \ell = m \neq 0, \\ \pi & \text{if } \ell = m = 0. \end{cases} \tag{1.6.15}$$

The Rodrigues formula for $T_m(t)$ has the form

$$T_m(t) = c\,(1-t^2)^{1/2} \left(\frac{d}{dt}\right)^m (1-t^2)^{m-1/2}, \qquad c = \frac{(-1)^m\, \pi^{1/2}}{2^m\, \Gamma(m+1/2)}. \tag{1.6.16}$$

As in (1.6.13),

$$|T_m(t)| \le c \begin{cases} 1 & \text{if } m \text{ is even}, \\ |t| & \text{if } m \text{ is odd}, \end{cases} \qquad c \equiv c(m) = \text{const}; \tag{1.6.17}$$

cf. 10.11(22) in Erdélyi [145]. The orthogonal systems of Jacobi, Gegenbauer, and Chebyshev polynomials are complete in the space $L^2[-1,1]; w)$ with the weight w, which is $w_{\alpha,\beta}(t)$, $w_\lambda(t)$, or $w_0(t)$, respectively.

The following equalities for the Mellin transforms are simple consequences of 47(1) and 48(4) from Marichev [396, Sec. 10 (10)]. Let $\eta = 0$ if m is even and $\eta = 1$ if m is odd,

$$c_{\lambda,m} = \frac{\Gamma(2\lambda + m)\, \Gamma(\lambda + 1/2)}{2m!\, \Gamma(2\lambda)}, \qquad \lambda > -1/2, \quad \lambda \neq 0. \tag{1.6.18}$$

Then[1]

$$\alpha_m(z) \equiv \int_0^1 u^{z-1}(1-u^2)^{\lambda-1/2}\, C_m^\lambda(u)\, du \tag{1.6.19}$$

$$= \frac{c_{\lambda,m}\, \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\lambda + \frac{z+1+m}{2}\right) \Gamma\left(\frac{z+1-m}{2}\right)}, \qquad Re\, z > -\eta;$$

$$\beta_m(z) \equiv \int_0^1 u^{z-1}(1-u^2)^{\lambda-1/2}\, C_m^\lambda(1/u)\, du \tag{1.6.20}$$

$$= \frac{c_{\lambda,m}\, \Gamma\left(\frac{z-m}{2}\right) \Gamma\left(\lambda + \frac{z+m}{2}\right)}{\Gamma\left(\lambda + \frac{z}{2}\right) \Gamma\left(\lambda + \frac{z+1}{2}\right)}, \qquad Re\, z > m.$$

[1] If m is odd, then $\eta = 1$ and the right-hand side of (1.6.19) is understood for $-1 < Re\, z \le 0$ by continuity (the same for (1.6.21)).

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These formulas can be equivalently written in a different form; see 2.21.2(5) and 2.21.2(25) in Prudnikov, Brychkov, Marichev [476]. Similarly, by 18(1) and 19(4) from Marichev [396, Sec. 10 (10)], we have

$$\int_0^1 u^{z-1}(1-u^2)^{-1/2}\, T_m(u)\, du = \frac{\pi^{1/2}\,\Gamma\left(\frac{z}{2}\right)\,\Gamma\left(\frac{z+1}{2}\right)}{2\,\Gamma\left(\frac{z+1+m}{2}\right)\,\Gamma\left(\frac{z+1-m}{2}\right)}, \quad (1.6.21)$$

$$\int_0^1 u^{z-1}(1-u^2)^{-1/2}\, T_m(1/u)\, du = \frac{\pi^{1/2}\,\Gamma\left(\frac{z-m}{2}\right)\,\Gamma\left(\frac{z+m}{2}\right)}{2\,\Gamma\left(\frac{z}{2}\right)\,\Gamma\left(\frac{z+1}{2}\right)}, \quad (1.6.22)$$

where $Re\, z > -\eta$ and $Re\, z > m$, respectively.

1.7 The Laplace Operator

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. The Laplace operator Δ acts on functions $f(x)$ by the formula

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}. \quad (1.7.1)$$

Proposition 1.5. *The operator Δ commutes with orthogonal transformations of $\mathbb{R}^n$, that is,*

$$\Delta(f \circ \rho) = \Delta f \circ \rho \quad \forall \rho \in O(n). \quad (1.7.2)$$

Proof. Let $\rho = (\rho_{ij})$. The rows of this matrix form an orthonormal basis of $\mathbb{R}^n$, and therefore

$$\sum_{j=1}^{n} \rho_{ij}\, \rho_{kj} = \delta_{ik} \qquad \text{(the Kronecker symbol).} \quad (1.7.3)$$

For $x \in \mathbb{R}^n$ we have $(\rho x)_i = \rho_{i1}x_1 + \cdots + \rho_{in}x_n$. Hence

$$\partial_j[f(\rho x)] = \sum_{i=1}^{n} \frac{\partial f(\rho x)}{\partial(\rho x)_i}\, \rho_{ij}, \qquad \partial_j^2 f(\rho x) = \sum_{i,k=1}^{n} \frac{\partial^2 f(\rho x)}{\partial(\rho x)_i\, \partial(\rho x)_k}\, \rho_{ij}\, \rho_{kj}.$$

Now summing over j and (1.7.3) give (1.7.2). □

If f is a radial function, that is, $f(x) = f_0(|x|)$ for some function f_0, then, by Proposition 1.5, Δf is also radial. A simple calculation gives

$$\Delta[f_0(|x|)] = (\Delta_{rad} f_0)(|x|), \qquad \Delta_{rad} = r^{1-n}\frac{\partial}{\partial r} r^{n-1}\frac{\partial}{\partial r}, \qquad r = |x| \neq 0. \quad (1.7.4)$$

The operator Δ_{rad} is called the *radial part* of the Laplace operator Δ.