Langlands Correspondence for Loop Groups

The Langlands Program was conceived initially as a bridge between Number Theory and Automorphic Representations, and has now expanded into such areas as Geometry and Quantum Field Theory, weaving together seemingly unrelated disciplines into a web of tantalizing conjectures. This book provides a new chapter in this grand project. It develops the geometric Langlands Correspondence for Loop Groups, a new approach, from a unique perspective offered by affine Kac–Moody algebras. The theory offers fresh insights into the world of Langlands dualities, with many applications to Representation Theory of Infinite-dimensional Algebras, and Quantum Field Theory. This introductory text builds the theory from scratch, with all necessary concepts defined and the essential results proved along the way. Based on courses taught by the author at Berkeley, the book provides many open problems which could form the basis for future research, and is accessible to advanced undergraduate students and beginning graduate students.

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EDWARD FRENKEL
University of California, Berkeley

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For my parents

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Concentric geometries of transparency slightly
joggled sink through algebras of proud

inwardlyness to collide spirally with iron arithmethics...

– E.E. Cummings, “W [ViVa]” (1931)

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Preface

The Langlands Program has emerged in recent years as a blueprint for a Grand Unified Theory of Mathematics. Conceived initially as a bridge between Number Theory and Automorphic Representations, it has now expanded into such areas as Geometry and Quantum Field Theory, weaving together seemingly unrelated disciplines into a web of tantalizing conjectures. The Langlands correspondence manifests itself in a variety of ways in these diverse areas of mathematics and physics, but the same salient features, such as the appearance of the Langlands dual group, are always present. This points to something deeply mysterious and elusive, and that is what makes this correspondence so fascinating.

One of the prevalent themes in the Langlands Program is the interplay between the *local* and *global* pictures. In the context of Number Theory, for example, "global" refers to a number field (a finite extension of the field of rational numbers) and its Galois group, while "local" means a local field, such as the field of p-adic numbers, together with its Galois group. On the other side of the Langlands correspondence we have, in the global case, automorphic representations, and, in the local case, representations of a reductive group, such as GL_n, over the local field.

In the geometric context the cast of characters changes: on the Galois side we now have vector *bundles with flat connection* on a complex Riemann surface X in the global case, and on the punctured disc $D^\times$ around a point of X in the local case. The definition of the objects on the other side of the geometric Langlands correspondence is more subtle. It is relatively well understood (after works of A. Beilinson, V. Drinfeld, G. Laumon and others) in the special case when the flat connection on our bundle has no singularities. Then the corresponding objects are the so-called "Hecke eigensheaves" on the moduli spaces of vector bundles on X. These are the geometric analogues of unramified automorphic functions. The unramified global geometric Langlands correspondence is then supposed to assign to a flat connection on our bundle (without singularities) a Hecke eigensheaf. (This is discussed in a recent review [F7], among other places, where we refer the reader for more details.)

However, in the more general case of *connections with ramification*, that is, with singularities, the geometric Langlands correspondence is much more mysterious, both in the local and in the global case. Actually, the impetus now shifts more to the local story. This is because the flat connections that we consider have finitely many singular points on our Riemann surface. The global ramified correspondence is largely determined by what happens on the punctured discs around those points, which is in the realm of the local correspondence. So the question really becomes: *what is the geometric analogue of the local Langlands correspondence?*

Now, the classical local Langlands correspondence relates representations of p-adic groups and Galois representations. In the geometric version we should replace a p-adic group by the (formal) *loop group* $G(\!(t)\!)$, the group of maps from the (formal) punctured disc $D^\times$ to a complex reductive algebraic group G. Galois representations should be replaced by vector bundles on $D^\times$ with a flat connection. These are the local geometric Langlands parameters. To each of them we should be able to attach a representation of the formal loop group.

Recently, Dennis Gaitsgory and I have made a general proposal describing these representations of loop groups. An important new element in our proposal is that, in contrast to the classical correspondence, the loop group now acts on categories rather than vector spaces. Thus, the Langlands correspondence for loop groups is categorical: we associate *categorical representations* of $G(\!(t)\!)$ to local Langlands parameters. We have proposed how to construct these categories using representations of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$, which is a central extension of the loop Lie algebra $\mathfrak{g}(\!(t)\!)$. Therefore the local geometric Langlands correspondence appears as the result of a successful marriage of the Langlands philosophy and the representation theory of affine Kac–Moody algebras.

Affine Kac–Moody algebras have a parameter, called the level. For a special value of this parameter, called the *critical level*, the completed enveloping algebra of an affine Kac–Moody algebra acquires an unusually large *center*. In 1991, Boris Feigin and I showed that this center is canonically isomorphic to the algebra of functions on the space of *opers* on $D^\times$. Opers are bundles on $D^\times$ with a flat connection and an additional datum (as defined by Drinfeld–Sokolov and Beilinson–Drinfeld). Remarkably, their structure group turns out to be not G, but the *Langlands dual group* ${}^LG$, in agreement with the general Langlands philosophy. This is the central result, which implies that the same salient features permeate both the representation theory of p-adic groups and the (categorical) representation theory of loop groups.

This result had been conjectured by V. Drinfeld, and it plays an important role in his and A. Beilinson's approach to the global geometric Langlands correspondence, via quantization of the Hitchin systems. The isomorphism between the center and functions on opers means that the category of representations of $\widehat{\mathfrak{g}}$ of critical level "lives" over the space of ${}^L G$-opers on $D^\times$, and the loop group $G((t))$ acts "fiberwise" on this category. In a nutshell, the proposal of Gaitsgory and myself is that the "fibers" of this category are the sought-after categorical representations of $G((t))$ corresponding to the local Langlands parameters underlying the ${}^L G$-opers. This has many non-trivial consequences, both for the local and global geometric Langlands correspondence, and for the representation theory of $\widehat{\mathfrak{g}}$. We hope that further study of these categories will give us new clues and insights into the mysteries of the Langlands correspondence.†

The goal of this book is to present a systematic and self-contained introduction to the local geometric Langlands correspondence for loop groups and the related representation theory of affine Kac–Moody algebras. It covers the research done in this area over the last twenty years and is partially based on the graduate courses that I have taught at UC Berkeley in 2002 and 2004. In the book, the entire theory is built from scratch, with all necessary concepts defined and all essential results proved along the way. We introduce such concepts as the Weil-Deligne group, Langlands dual group, affine Kac–Moody algebras, vertex algebras, jet schemes, opers, Miura opers, screening operators, etc., and illustrate them by detailed examples. In particular, many of the results are first explained in the simplest case of SL_2. Practically no background beyond standard college algebra is required from the reader (except possibly in the last chapter); we even explain some standard notions, such as universal enveloping algebras, in the Appendix.

In the opening chapter, we present a pedagogical overview of the classical Langlands correspondence and a motivated step-by-step passage to the geometric setting. This leads us to the study of affine Kac–Moody algebras and in particular the center of the completed enveloping algebra. We then review in great detail the construction of a series of representations of affine Kac–Moody algebras, called *Wakimoto modules*. They were defined by Feigin and myself in the late 1980s following the work of M. Wakimoto. These modules give us an effective tool for developing the representation theory of affine algebras. In particular, they are crucial in our proof of the isomorphism between the spectrum of the center and opers. A detailed exposition of the Wakimoto modules and the proof of this isomorphism constitute the main part of this book. These results allow us to establish a deep link between the representation theory of affine Kac–Moody algebras of critical level and the geometry of opers. In the closing chapter, we review the results and conjectures of

† We note that A. Beilinson has another proposal [Bei] for the local geometric Langlands correspondence, using representations of affine Kac–Moody algebras of integral levels less than critical. It would be interesting to understand the connection between his proposal and ours.

Gaitsgory and myself describing the representation categories associated to opers in the framework of the Langlands correspondence. I also discuss the implications of this for the global geometric Langlands correspondence. These are only the first steps of a new theory, which we hope will ultimately help us reveal the secrets of Langlands duality.

Contents

Here is a more detailed description of the contents of this book.

Chapter 1 is the introduction to the subject matter of this book. We begin by giving an overview of the local and global Langlands correspondence in the classical setting. Since the global case is discussed in great detail in my recent review [F7], I concentrate here mostly on the local case. Next, I explain what changes one needs to make in order to transport the local Langlands correspondence to the realm of geometry and the representation theory of loop groups. I give a pedagogical account of Galois groups, principal bundles with connections and central extensions, among other topics. This discussion leads us to the following question: how to attach to each local geometric Langlands parameter an abelian category equipped with an action of the formal loop group?

In Chapter 2 we take up this question in the context of the representation theory of affine Kac–Moody algebras. This motivates us to study the center of the completed enveloping algebra of $\widehat{\mathfrak{g}}$. First, we do that by elementary means, but very quickly we realize that we need a more sophisticated technique. This technique is the theory of vertex algebras. We give a crash course on vertex algebras (following [FB]), summarizing all necessary concepts and results.

Armed with these results, we begin in Chapter 3 a more in-depth study of the center of the completed enveloping algebra of $\widehat{\mathfrak{g}}$ at the critical level (we find that the center is trivial away from the critical level). We describe the center in the case of the simplest affine Kac–Moody algebra $\widehat{\mathfrak{sl}}_2$ and the quasi-classical analogue of the center for an arbitrary $\widehat{\mathfrak{g}}$.

In Chapter 4 we introduce the key geometric concept of opers, introduced originally in [DS, BD1]. We state the main result, due to Feigin and myself [FF6, F4], that the center at the critical level, corresponding to $\widehat{\mathfrak{g}}$, is isomorphic to the algebra of functions on ^LG-opers.

In order to prove this result, we need to develop the theory of Wakimoto modules. This is done in Chapters 5 and 6, following [F4]. We start by explaining the analogous theory for finite-dimensional simple Lie algebras, which serves as a prototype for our construction. Then we explain the non-trivial elements of the infinite-dimensional case, such as the cohomological obstruction to realizing a loop algebra in the algebra of differential operators on a loop space. This leads to a conceptual explanation of the non-triviality of the critical level. In Chapter 6 we complete the construction of Wakimoto modules, both at the critical and non-critical levels. We prove some useful results on representations of affine Kac–Moody algebras, such as the Kac-Kazhdan conjecture.

Having built the theory of Wakimoto modules, we are ready to tackle the isomorphism between the center and the algebra of functions on opers. At the beginning of Chapter 7 we give a detailed overview of the proof of this isomorphism. In the rest of Chapter 7 we introduce an important class of intertwining operators between Wakimoto modules called the screening operators. We use these operators and some results on associated graded algebras in Chapter 8 to complete the proof of our main result and to identify the center with functions on opers (here we follow [F4]). In particular, we clarify the origins of the appearance of the Langlands dual group in this isomorphism, tracing it back to a certain duality between vertex algebras known as the $\mathcal{W}$-algebras. At the end of the chapter we discuss the vertex Poisson structure on the center and identify the action of the center on Wakimoto modules with the Miura transformation.

In Chapter 9 we undertake a more in-depth study of representations of affine Kac–Moody algebras of critical level. We first introduce (following [BD1] and [FG2]) certain subspaces of the space of opers on $D^\times$: opers with regular singularities and nilpotent opers, and explain the interrelations between them. We then discuss Miura opers with regular singularities and the action of the Miura transformation on them, following [FG2]. Finally, we describe the results of [FG2] and [FG6] on the algebras of endomorphisms of the Verma modules and the Weyl modules of critical level.

In Chapter 10 we bring together the results of this book to explain the proposal for the local geometric Langlands correspondence made by Gaitsgory and myself. We review the results and conjectures of our works [FG1]–[FG6], emphasizing the analogies between the geometric and the classical Langlands correspondence. We discuss in detail the interplay between opers and local systems. We then consider the simplest local system; namely, the trivial one. The corresponding categorical representations of $G((t))$ are the analogues of unramified representations of p-adic groups. Already in this case we will see rather non-trivial elements, which emulate the corresponding elements of the classical theory and at the same time generalize them in a non-trivial way. The next, and considerably more complicated, example is that of local systems on $D^\times$ with regular singularity and unipotent monodromy. These are the analogues of the tamely ramified representations of the Galois group of a p-adic field. The corresponding categories turn out to be closely related to categories of quasicoherent sheaves on the Springer fibers, which are algebraic subvarieties of the flag variety of the Langlands dual group. We summarize the conjectures and results of [FG2] concerning these categories and illustrate them by explicit computations in the case of $\widehat{\mathfrak{sl}}_2$. We also formulate some open problems in this direction. Finally, we discuss the implications of this approach for the global Langlands correspondence.

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