1
Local Langlands correspondence

In this introductory chapter we explain in detail what we mean by "local Langlands correspondence for loop groups." We begin by giving a brief overview of the local Langlands correspondence for reductive groups over local non-archimedian fields, such as $\mathbb{F}_q((t))$, the field of Laurent power series over a finite field $\mathbb{F}_q$. We wish to understand an analogue of this correspondence when $\mathbb{F}_q$ is replaced by the field $\mathbb{C}$ of complex numbers. The role of the reductive group will then be played by the formal loop group $G(\mathbb{C}((t)))$. We discuss, following [FG2], how the setup of the Langlands correspondence should change in this case. This discussion will naturally lead us to categories of representations of the corresponding affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ equipped with an action of $G(\mathbb{C}((t)))$, the subject that we will pursue in the rest of this book.

1.1 The classical theory

The local Langlands correspondence relates smooth representations of reductive algebraic groups over local fields and representations of the Galois group of this field. We start out by defining these objects and explaining the main features of this correspondence. As the material of this section serves motivational purposes, we will only mention those aspects of this story that are most relevant for us. For a more detailed treatment, we refer the reader to the informative surveys [V, Ku] and references therein.

1.1.1 Local non-archimedian fields

Let F be a local non-archimedian field. In other words, F is the field $\mathbb{Q}_p$ of p-adic numbers, or a finite extension of $\mathbb{Q}_p$, or F is the field $\mathbb{F}_q((t))$ of formal Laurent power series with coefficients in $\mathbb{F}_q$, the finite field with q elements.

We recall that for any q of the form p^n, where p is a prime, there is a unique, up to isomorphism, finite field of characteristic p with q elements. An element of $\mathbb{F}_q((t))$ is an expression of the form

$$\sum_{n \in \mathbb{Z}} a_n t^n, \qquad a_n \in \mathbb{F}_q,$$

such that $a_n = 0$ for all n less than some integer N. In other words, these are power series infinite in the positive direction and finite in the negative direction. Recall that a p-adic number may also be represented by a series

$$\sum_{n \in \mathbb{Z}} b_n p^n, \qquad b_n \in \{0, 1, \ldots, p-1\},$$

such that $b_n = 0$ for all n less than some integer N. We see that elements of $\mathbb{Q}_p$ look similar to elements of $\mathbb{F}_p((t))$. Both fields are complete with respect to the topology defined by the norm taking value α^{-N} on the above series if $a_N \neq 0$ and $a_n = 0$ for all $n < N$, where α is a fixed positive real number between 0 and 1. But the laws of addition and multiplication in the two fields are different: with "carry" to the next digit in the case of $\mathbb{Q}_p$, but without "carry" in the case of $\mathbb{F}_p((t))$. In particular, $\mathbb{Q}_p$ has characteristic 0, while $\mathbb{F}_q((t))$ has characteristic p. More generally, elements of a finite extension of $\mathbb{Q}_p$ look similar to elements of $\mathbb{F}_q((t))$ for some $q = p^n$, but, again, the rules of addition and multiplication, as well as their characteristics, are different.

1.1.2 Smooth representations of $GL_n(F)$

Now consider the group $GL_n(F)$, where F is a local non-archimedian field. A representation of $GL_n(F)$ on a complex vector space V is a homomorphism $\pi : GL_n(F) \to \operatorname{End} V$ such that $\pi(gh) = \pi(g)\pi(h)$ and $\pi(1) = \mathrm{Id}$. Define a topology on $GL_n(F)$ by stipulating that the base of open neighborhoods of $1 \in GL_n(F)$ is formed by the congruence subgroups $K_N, N \in \mathbb{Z}_+$. In the case when $F = \mathbb{F}_q((t))$, the group K_N is defined as follows:

$$K_N = \{g \in GL_n(\mathbb{F}_q[[t]]) \,|\, g \equiv 1 \mod t^N\},$$

and for $F = \mathbb{Q}_p$ it is defined in a similar way. For each $v \in V$ we obtain a map $\pi(\cdot)v : GL_n(F) \to V, g \mapsto \pi(g)v$. A representation (V, π) is called **smooth** if the map $\pi(\cdot)v$ is continuous for each v, where we give V the discrete topology. In other words, V is smooth if for any vector $v \in V$ there exists $N \in \mathbb{Z}_+$ such that

$$\pi(g)v = v, \qquad \forall g \in K_N.$$

We are interested in describing the equivalence classes of irreducible smooth representations of $GL_n(F)$. Surprisingly, those turn out to be related to objects of a different kind: n-dimensional representations of the Galois group of F.

1.1.3 The Galois group

Suppose that F is a subfield of K. Then the **Galois group** $\mathrm{Gal}(K/F)$ consists of all automorphisms σ of the field K such that $\sigma(y) = y$ for all $y \in F$.

Let F be a field. The algebraic closure of F is a field obtained by adjoining to F the roots of all polynomials with coefficients in F. In the case when $F = \mathbb{F}_q((t))$ some of the extensions of F may be non-separable. An example of such an extension is the field $\mathbb{F}_q((t^{1/p}))$. The polynomial defining this extension is $x^p - t$, but in $\mathbb{F}_q((t^{1/p}))$ it has multiple roots because

$$x^p - (t^{1/p})^p = (x - t^{1/p})^p.$$

The Galois group $\mathrm{Gal}(\mathbb{F}_q((t^{1/p})), \mathbb{F}_q((t)))$ of this extension is trivial, even though the degree of the extension is p.

This extension should be contrasted to the separable extensions $\mathbb{F}_q((t^{1/n}))$, where n is not divisible by p. This extension is defined by the polynomial $x^n - t$, which now has no multiple roots:

$$x^n - t = \prod_{i=0}^{n-1}(x - \zeta^i t^{1/n}),$$

where ζ is a primitive nth root of unity in the algebraic closure of $\mathbb{F}_q$. The corresponding Galois group is identified with the group $(\mathbb{Z}/n\mathbb{Z})^\times$, the group of invertible elements of $\mathbb{Z}/n\mathbb{Z}$.

We wish to avoid the non-separable extensions, because they do not contribute to the Galois group. (There are no non-separable extensions if F has characteristic zero, e.g., for $F = \mathbb{Q}_p$.) Let $\overline{F}$ be the maximal separable extension inside a given algebraic closure of F. It is uniquely defined up to isomorphism.

Let $\mathrm{Gal}(\overline{F}/F)$ be the **absolute Galois group** of F. Its elements are the automorphisms σ of the field $\overline{F}$ such that $\sigma(y) = y$ for all $y \in F$.

To gain some experience with Galois groups, let us look at the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Here $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$, which can be defined as the inductive limit of the fields $\mathbb{F}_{q^N}, N \in \mathbb{Z}_+$, with respect to the natural embeddings $\mathbb{F}_{q^N} \hookrightarrow \mathbb{F}_{q^M}$ for N dividing M. Therefore $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is isomorphic to the inverse limit of the Galois groups $\mathrm{Gal}(\mathbb{F}_{q^N}/\mathbb{F}_q)$ with respect to the natural surjections

$$\mathrm{Gal}(\mathbb{F}_{q^M}/\mathbb{F}_q) \twoheadrightarrow \mathrm{Gal}(\mathbb{F}_{q^N}/\mathbb{F}_q), \qquad \forall N|M.$$

The group $\mathrm{Gal}(\mathbb{F}_{q^N}/\mathbb{F}_q)$ is easy to describe: it is generated by the **Frobenius automorphism** $x \mapsto x^q$ (note that it stabilizes $\mathbb{F}_q$), which has order N, so that $\mathrm{Gal}(\mathbb{F}_{q^N}/\mathbb{F}_q) \simeq \mathbb{Z}/N\mathbb{Z}$. Therefore we find that

$$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \simeq \widehat{\mathbb{Z}} \stackrel{\text{def}}{=} \varprojlim \mathbb{Z}/N\mathbb{Z},$$

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where we have the surjective maps $\mathbb{Z}/M\mathbb{Z} \twoheadrightarrow \mathbb{Z}/N\mathbb{Z}$ for $N|M$. The group $\widehat{\mathbb{Z}}$ contains $\mathbb{Z}$ as a subgroup.

Let $F = \mathbb{F}_q((t))$. Observe that we have a natural map $\mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ obtained by applying an automorphism of $\overline{F}$ to $\overline{\mathbb{F}}_q \subset \overline{F}$. A similar map also exists when F has characteristic 0. Let W_F be the preimage of the subgroup $\mathbb{Z} \subset \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. This is the **Weil group** of F. Let ν be the corresponding homomorphism $W_F \to \mathbb{Z}$. Let $W'_F = W_F \ltimes \mathbb{C}$ be the semi-direct product of W_F and the one-dimensional complex additive group $\mathbb{C}$, where W_F acts on $\mathbb{C}$ by the formula

$$\sigma x \sigma^{-1} = q^{\nu(\sigma)} x, \qquad \sigma \in W_F, x \in \mathbb{C}. \tag{1.1.1}$$

This is the **Weil–Deligne group** of F.

An n-dimensional complex representation of W'_F is by definition a homomorphism $\rho' : W'_F \to GL_n(\mathbb{C})$, which may be described as a pair (ρ, u), where ρ is an n-dimensional representation of W_F, $u \in \mathfrak{gl}_n(\mathbb{C})$, and we have $\rho(\sigma)u\rho(\sigma)^{-1} = q^{\nu(\sigma)}u$ for all $\sigma \in W_F$. The group W_F is topological, with respect to the Krull topology (in which the open neighborhoods of the identity are the normal subgroups of finite index). The representation (ρ, u) is called **admissible** if ρ is continuous (equivalently, factors through a finite quotient of W_F) and semisimple, and u is a nilpotent element of $\mathfrak{gl}_n(\mathbb{C})$.

The group W'_F was introduced by P. Deligne [De2]. The idea is that by adjoining the nilpotent element u to W_F we obtain a group whose complex admissible representations are the same as continuous ℓ-adic representations of W_F (where $\ell \neq p$ is a prime).

1.1.4 The local Langlands correspondence for GL_n

Now we are ready to state the local Langlands correspondence for the group GL_n over a local non-archimedian field F. It is a bijection between two different sorts of data. One is the set of equivalence classes of irreducible smooth representations of $GL_n(F)$. The other is the set of equivalence classes of n-dimensional admissible representations of W'_F. We represent it schematically as follows:

n-dimensional admissible representations of W'_F	$\Longleftrightarrow$	irreducible smooth representations of $GL_n(F)$

This correspondence is supposed to satisfy an overdetermined system of constraints which we will not recall here (see, e.g., [Ku]).

The local Langlands correspondence for GL_n is a theorem. In the case when $F = \mathbb{F}_q((t))$ it has been proved in [LRS], and when $F = \mathbb{Q}_p$ or its finite extension in [HT] and also in [He]. We refer the readers to these papers and to the review [C] for more details.

Despite an enormous effort made in the last two decades to understand it, the local Langlands correspondence still remains a mystery. We do know that the above bijection exists, but we cannot yet explain in a completely satisfactory way *why* it exists. We do not know the deep underlying reasons that make such a correspondence possible. One way to try to understand it is to see how general it is. In the next section we will discuss one possible generalization of this correspondence, where we replace the group GL_n by an arbitrary reductive algebraic group defined over F.

1.1.5 Generalization to other reductive groups

Let us replace the group GL_n by an arbitrary connected reductive group G over a local non-archimedian field F. The group $G(F)$ is also a topological group, and there is a notion of smooth representation of $G(F)$ on a complex vector space. It is natural to ask whether we can relate irreducible smooth representations of $G(F)$ to representations of the Weil–Deligne group W'_F. This question is addressed in the general local Langlands conjectures. It would take us too far afield to try to give here a precise formulation of these conjectures. So we will only indicate some of the objects involved, referring the reader to the articles [V, Ku], where these conjectures are described in great detail.

Recall that in the case when $G = GL_n$ the irreducible smooth representations are parametrized by admissible homomorphisms $W'_F \to GL_n(\mathbb{C})$. In the case of a general reductive group G, the representations are conjecturally parametrized by admissible homomorphisms from W'_F to the so-called Langlands dual group ${}^LG$, which is defined over $\mathbb{C}$.

In order to explain the notion of the Langlands dual group, consider first the group G over the closure $\overline{F}$ of the field F. All maximal tori T of this group are conjugate to each other and are necessarily split, i.e., we have an isomorphism $T(\overline{F}) \simeq (\overline{F}^\times)^n$. For example, in the case of GL_n, all maximal tori are conjugate to the subgroup of diagonal matrices. We associate to $T(\overline{F})$ two lattices: the weight lattice $X^*(T)$ of homomorphisms $T(\overline{F}) \to \overline{F}^\times$ and the coweight lattice $X_*(T)$ of homomorphisms $\overline{F}^\times \to T(\overline{F})$. They contain the sets of roots $\Delta \subset X^*(T)$ and coroots $\Delta^\vee \subset X_*(T)$, respectively. The quadruple $(X^*(T), X_*(T), \Delta, \Delta^\vee)$ is called the root data for G over $\overline{F}$. The root data determines G up to an isomorphism defined over $\overline{F}$. The choice of a Borel subgroup $B(\overline{F})$ containing $T(\overline{F})$ is equivalent to a choice of a basis in Δ; namely, the set of simple roots Δ_s, and the corresponding basis $\Delta_s^\vee$ in $\Delta^\vee$.

Now given $\gamma \in \mathrm{Gal}(\overline{F}/F)$, there is $g \in G(\overline{F})$ such that $g(\gamma(T(\overline{F}))g^{-1} = T(\overline{F})$ and $g(\gamma(B(\overline{F}))g^{-1} = B(\overline{F})$. Then g gives rise to an automorphism of the based root data $(X^*(T), X_*(T), \Delta_s, \Delta_s^\vee)$. Thus, we obtain an action of $\mathrm{Gal}(\overline{F}/F)$ on the based root data.

Let us now exchange the lattices of weights and coweights and the sets of simple roots and coroots. Then we obtain the based root data

$$(X_*(T), X^*(T), \Delta_s^\vee, \Delta_s)$$

of a reductive algebraic group over $\mathbb{C}$, which is denoted by ${}^L G^\circ$. For instance, the group GL_n is self-dual, the dual of SO_{2n+1} is Sp_{2n}, the dual of Sp_{2n} is SO_{2n+1}, and SO_{2n} is self-dual.

The action of $\mathrm{Gal}(\overline{F}/F)$ on the based root data gives rise to its action on ${}^L G^\circ$. The semi-direct product ${}^L G = \mathrm{Gal}(\overline{F}/F) \ltimes {}^L G^\circ$ is called the **Langlands dual group** of G.

The local Langlands correspondence for the group $G(F)$ relates the equivalence classes of irreducible smooth representations of $G(F)$ to the equivalence classes of admissible homomorphisms $W'_F \to {}^L G$. However, in general this correspondence is much more subtle than in the case of GL_n. In particular, we need to consider simultaneously representations of all inner forms of G, and a homomorphism $W'_F \to {}^L G$ corresponds in general not to a single irreducible representation of $G(F)$, but to a finite set of representations called an L-**packet**. To distinguish between them, we need additional data (see [V] for more details; some examples are presented in Section 10.4.1 below). But in the first approximation we can say that the essence of the local Langlands correspondence is that

irreducible smooth representations of $G(F)$ are parameterized in terms of admissible homomorphisms $W'_F \to {}^L G$.

1.1.6 On the global Langlands correspondence

We close this section with a brief discussion of the global Langlands correspondence and its connection to the local one. We will return to this subject in Section 10.5.

Let X be a smooth projective curve over $\mathbb{F}_q$. Denote by F the field $\mathbb{F}_q(X)$ of rational functions on X. For any closed point x of X we denote by F_x the completion of F at x and by $\mathcal{O}_x$ its ring of integers. If we choose a local coordinate t_x at x (i.e., a rational function on X which vanishes at x to order one), then we obtain isomorphisms $F_x \simeq \mathbb{F}_{q_x}((t_x))$ and $\mathcal{O}_x \simeq \mathbb{F}_{q_x}[[t_x]]$, where $\mathbb{F}_{q_x}$ is the residue field of x; in general, it is a finite extension of $\mathbb{F}_q$ containing $q_x = q^{\deg(x)}$ elements.

Thus, we now have a local field attached to each point of X. The ring $\mathbb{A} = \mathbb{A}_F$ of **adèles** of F is by definition the *restricted* product of the fields F_x, where x runs over the set $|X|$ of all closed points of X. The word "restricted" means that we consider only the collections $(f_x)_{x \in |X|}$ of elements of F_x in which $f_x \in \mathcal{O}_x$ for all but finitely many x. The ring $\mathbb{A}$ contains the field F, which is embedded into $\mathbb{A}$ diagonally, by taking the expansions of rational functions on X at all points.

While in the local Langlands correspondence we considered irreducible smooth representations of the group GL_n over a local field, in the global Langlands correspondence we consider irreducible **automorphic representations** of the group $GL_n(\mathbb{A})$. The word "automorphic" means, roughly, that the representation may be realized in a reasonable space of functions on the quotient $GL_n(F)\backslash GL_n(\mathbb{A})$ (on which the group $GL_n(\mathbb{A})$ acts from the right).

On the other side of the correspondence we consider n-dimensional representations of the Galois group $\mathrm{Gal}(\overline{F}/F)$, or, more precisely, the Weil group W_F, which is a subgroup of $\mathrm{Gal}(\overline{F}/F)$ defined in the same way as in the local case.

Roughly speaking, the global Langlands correspondence is a bijection between the set of equivalence classes of n-dimensional representations of W_F and the set of equivalence classes of irreducible automorphic representations of $GL_n(\mathbb{A})$:

n-dimensional representations of W_F	$\Longleftrightarrow$	irreducible automorphic representations of $GL_n(\mathbb{A})$

The precise statement is more subtle. For example, we should consider the so-called ℓ-adic representations of the Weil group (while in the local case we considered the admissible complex representations of the Weil–Deligne group; the reason is that in the local case those are equivalent to the ℓ-adic representations). Moreover, under this correspondence important invariants attached to the objects appearing on both sides (Frobenius eigenvalues on the Galois side and the Hecke eigenvalues on the other side) are supposed to match. We refer the reader to Part I of the review [F7] for more details.

The global Langlands correspondence has been proved for GL_2 in the 1980's by V. Drinfeld [Dr1]–[Dr4] and more recently by L. Lafforgue [Laf] for GL_n with an arbitrary n.

The global and local correspondences are compatible in the following sense. We can embed the Weil group W_{F_x} of each of the local fields F_x into the global Weil group W_F. Such an embedding is not unique, but it is well-defined up to conjugation in W_F. Therefore an equivalence class σ of n-dimensional representations of W_F gives rise to a well-defined equivalence class σ_x of n-dimensional representations of W_{F_x} for each $x \in X$. By the local Langlands correspondence, to σ_x we can attach an equivalence class of irreducible smooth representations of $GL_n(F_x)$. Choose a representation π_x in this equivalence class. Then the automorphic representation of $GL_n(\mathbb{A})$ corresponding to σ

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is isomorphic to the restricted tensor product $\bigotimes'_{x \in X} \pi_x$. This is a very non-trivial statement, because a priori it is not clear why this tensor product may be realized in the space of functions on the quotient $GL_n(F)\backslash GL_n(\mathbb{A})$.

As in the local story, we may also wish to replace the group GL_n by an arbitrary reductive algebraic group defined over F. The general global Langlands conjecture predicts, roughly speaking, that irreducible automorphic representations of $G(\mathbb{A})$ are related to homomorphisms $W_F \to {}^L G$. But, as in the local case, the precise formulation of the conjecture for a general reductive group is much more intricate (see [Art]).

Finally, the global Langlands conjectures can also be stated over number fields (where they in fact originated). Then we take as the field F a finite extension of the field $\mathbb{Q}$ of rational numbers. Consider for example the case of $\mathbb{Q}$ itself. It is known that the completions of $\mathbb{Q}$ are (up to isomorphism) the fields of p-adic numbers $\mathbb{Q}_p$ for all primes p (non-archimedian) and the field $\mathcal{R}$ of real numbers (archimedian). So the primes play the role of points of an algebraic curve over a finite field (and the archimedian completion corresponds to an infinite point, in some sense). The ring of adèles $\mathbb{A}_{\mathbb{Q}}$ is defined in the same way as in the function field case, and so we can define the notion of an automorphic representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ or a more general reductive group. Conjecturally, to each equivalence class of n-dimensional representations of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we can attach an equivalence class of irreducible automorphic representations of $GL_n(\mathbb{A}_{\mathbb{Q}})$, but this correspondence is not expected to be a bijection because in the number field case it is known that some of the automorphic representations do not correspond to any Galois representations.

The Langlands conjectures in the number field case lead to very important and unexpected results. Indeed, many interesting representations of Galois groups can be found in "nature". For example, the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ will act on the geometric invariants (such as the étale cohomologies) of an algebraic variety defined over $\mathbb{Q}$. Thus, if we take an elliptic curve E over $\mathbb{Q}$, then we will obtain a two-dimensional Galois representation on its first étale cohomology. This representation contains a lot of important information about the curve E, such as the number of points of E over $\mathbb{Z}/p\mathbb{Z}$ for various primes p. The Langlands correspondence is supposed to relate these Galois representations to automorphic representations of $GL_2(\mathbb{A}_F)$ in such a way that the data on the Galois side, like the number of points of $E(\mathbb{Z}/p\mathbb{Z})$, are translated into something more tractable on the automorphic side, such as the coefficients in the q-expansion of the modular forms that encapsulate automorphic representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$. This leads to some startling consequences, such as the Taniyama-Shimura conjecture. For more on this, see [F7] and references therein.

The Langlands correspondence has proved to be easier to study in the function field case. The main reason is that in the function field case we can use the geometry of the underlying curve and various moduli spaces associated to this curve. A curve can also be considered over the field of complex numbers. Some recent results show that a version of the global Langlands correspondence also exists for such curves. The local counterpart of this correspondence is the subject of this book.

1.2 Langlands parameters over the complex field

We now wish to find a generalization of the local Langlands conjectures in which we replace the field $F = \mathbb{F}_q((t))$ by the field $\mathbb{C}((t))$. We would like to see how the ideas and patterns of the Langlands correspondence play out in this new context, with the hope of better understanding the deep underlying structures behind this correspondence.

So from now on G will be a connected reductive group over $\mathbb{C}$, and $G(F)$ the group $G((t)) = G(\mathbb{C}((t)))$, also known as the **loop group**; more precisely, the formal loop group, with the word "formal" referring to the algebra of formal Laurent power series $\mathbb{C}((t))$ (as opposed to the group $G(\mathbb{C}[t, t^{-1}])$, where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials, which may be viewed as the group of maps from the unit circle $|t| = 1$ to G, or "loops" in G).

Thus, we wish to study smooth representations of the loop group $G((t))$ and try to relate them to some "Langlands parameters," which we expect, by analogy with the case of local non-archimedian fields described above, to be related to the Galois group of $\mathbb{C}((t))$ and the Langlands dual group ${}^L G$.

The local Langlands correspondence for loop groups that we discuss in this book may be viewed as the first step in carrying the ideas of the Langlands Program to the realm of complex algebraic geometry. In particular, it has far-reaching consequences for the global geometric Langlands correspondence (see Section 10.5 and [F7] for more details). This was in fact one of the motivations for this project.

1.2.1 *The Galois group and the fundamental group*

We start by describing the Galois group $\mathrm{Gal}(\overline{F}/F)$ for $F = \mathbb{C}((t))$. Observe that the algebraic closure $\overline{F}$ of F is isomorphic to the inductive limit of the fields $\mathbb{C}((t^{1/n})), n \geq 0$, with respect to the natural inclusions $\mathbb{C}((t^{1/n})) \hookrightarrow \mathbb{C}((t^{1/m}))$ for n dividing m. Hence $\mathrm{Gal}(\overline{F}/F)$ is the inverse limit of the Galois groups

$$\mathrm{Gal}(\mathbb{C}((t^{1/n}))/C((t))) \simeq \mathbb{Z}/n\mathbb{Z},$$

where $\overline{k} \in \mathbb{Z}/n\mathbb{Z}$ corresponds to the automorphism of $\mathbb{C}((t^{1/n}))$ sending $t^{1/n}$ to $e^{2\pi i k/n} t^{1/n}$. The result is that

$$\mathrm{Gal}(\overline{F}/F) \simeq \widehat{\mathbb{Z}},$$

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where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$ that we have encountered before in Section 1.1.3.

Note, however, that in our study of the Galois group of $\mathbb{F}_q((t))$ the group $\widehat{\mathbb{Z}}$ appeared as its quotient corresponding to the Galois group of the field of coefficients $\mathbb{F}_q$. Now the field of coefficients is $\mathbb{C}$, hence algebraically closed, and $\widehat{\mathbb{Z}}$ is the *entire* Galois group of $\mathbb{C}((t))$.

The naive analogue of the Langlands parameter would be an equivalence class of homomorphisms $\mathrm{Gal}(\overline{F}/F) \to {}^LG$, i.e., a homomorphism $\widehat{\mathbb{Z}} \to {}^LG$. Since G is defined over $\mathbb{C}$ and hence all of its maximal tori are split, the group $G((t))$ also contains a split torus $T((t))$, where T is a maximal torus of G (but it also contains non-split maximal tori, as the field $\mathbb{C}((t))$ is not algebraically closed). Therefore the Langlands dual group ${}^LG$ is the direct product of the Galois group and the group ${}^LG^\circ$. Because it is a direct product, we may, and will, restrict our attention to ${}^LG^\circ$. In order to simplify our notation, from now on we will denote ${}^LG^\circ$ simply by ${}^LG$.

A homomorphism $\widehat{\mathbb{Z}} \to {}^LG$ necessarily factors through a finite quotient $\widehat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$. Therefore the equivalence classes of homomorphisms $\widehat{\mathbb{Z}} \to {}^LG$ are the same as the conjugacy classes of ${}^LG$ of finite order. There are too few of these to have a meaningful generalization of the Langlands correspondence. Therefore we look for a more sensible alternative.

Let us recall the connection between Galois groups and fundamental groups. Let X be an algebraic variety over $\mathbb{C}$. If $Y \to X$ is a covering of X, then the field $\mathbb{C}(Y)$ of rational functions on Y is an extension of the field $F = \mathbb{C}(X)$ of rational functions on X. The deck transformations of the cover, i.e., automorphisms of Y which induce the identity on X, give rise to automorphisms of the field $\mathbb{C}(Y)$ preserving $\mathbb{C}(X) \subset \mathbb{C}(Y)$. Hence we identify the Galois group $\mathrm{Gal}(\mathbb{C}(Y)/\mathbb{C}(X))$ with the group of deck transformations. If our cover is unramified, then this group may be identified with a quotient of the fundamental group of X. Otherwise, this group is isomorphic to a quotient of the fundamental group of X with the ramification divisor thrown out.

In particular, we obtain that the Galois group of the maximal unramified extension of $\mathbb{C}(X)$ (which we can view as the field of functions of the "maximal unramified cover" of X) is the profinite completion of the fundamental group $\pi_1(X)$ of X. Likewise, for any divisor $D \subset X$ the Galois group of the maximal extension of $\mathbb{C}(X)$ unramified away from D is the profinite completion of $\pi_1(X \backslash D)$. We denote it by $\pi_1^{\mathrm{alg}}(X \backslash D)$. The algebraic closure of $\mathbb{C}(X)$ is the inductive limit of the fields of functions on the maximal covers of X ramified at various divisors $D \subset X$ with respect to natural inclusions corresponding to the inclusions of the divisors. Hence the Galois group $\mathrm{Gal}(\overline{\mathbb{C}(X)}/\mathbb{C}(X))$ is the inverse limit of the groups $\pi_1^{\mathrm{alg}}(X \backslash D)$ with respect to the maps $\pi_1^{\mathrm{alg}}(X \backslash D') \to \pi_1^{\mathrm{alg}}(X \backslash D)$ for $D \subset D'$.

Strictly speaking, in order to define the fundamental group of X we need to pick a reference point $x \in X$. In the above discussion we have tacitly