

## PHYSICS OF SOLITONS

Solitons are waves with exceptional stability properties which appear in many areas of physics, from hydrodynamic tsunamis and fibre optic communications to solid state physics and the dynamics of biological molecules. Since they were first observed in 1834 they have fascinated scientists, not only for their spectacular experimental properties and the remarkable mathematical theories that they have initiated, but also for the new insight that they provide into many physical problems.

The basic properties of solitons are introduced here using examples from macroscopic physics such as blood pressure pulses and fibre optic communications. The book then presents the main theoretical methods and discusses a wide range of applications in detail. These applications include examples from solid state and atomic physics, for example, excitations in spin chains, conducting polymers and Bose–Einstein condensates and also biological physics (e.g. energy transfer in proteins and DNA fluctuations).

In addition to knowledge on the physics of solitons, the authors aim to familiarise the reader with a new way of thinking in physics. Instead of linear approximations followed by a perturbative approach to nonlinearities it is often more efficient to treat nonlinearities intrinsically, and to base the analysis on one of the soliton equations introduced in this book. This modelling process is stressed throughout the book and also discussed in Chapter 4.

Based on the authors' graduate course, this textbook gives an instructive view of the physics of solitons for students with a basic knowledge of general physics, and classical and quantum mechanics.

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THIERRY DAUXOIS AND MICHEL PEYRARD  
*Ecole Normale Supérieure de Lyon*



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## Preface

Since the first observation of a *soliton* by John Scott Russell in 1834, these exceptionally stable solitary waves have fascinated scientists for their spectacular experimental properties and their obvious elegance. But it is perhaps the remarkable mathematical properties of integrable systems having soliton solutions which attracted most of the attention. Mathematical aspects have been put forward in most of the books dealing with solitons because they lead to beautiful theories, such as the inverse scattering transform which derives the solution of a complex *nonlinear* equation from a series of steps which are all *linear* (see Chapter 7).

However, besides mathematics, the *physics* of solitons is also very fascinating, and at the heart of modern research. For instance many experiments on Bose–Einstein condensation, for which the 2001 Nobel prize in physics was awarded, are analysed in terms of the nonlinear Schrödinger equation, introduced in Chapter 3, which is one of the basic equations of soliton theory. The role of solitons in the physics of Bose–Einstein condensates is discussed in Chapter 14. The 2000 Nobel prize in chemistry, awarded to Heeger, MacDiarmid and Shirakawa, is also closely related to solitons because the charge carriers in conducting polymers are solitons. Chapter 13, which explains these phenomena, is based on a paper by Su, Schrieffer and Heeger.

Thus the physics of solitons is a very active research topic, to which we have contributed. However this book is not a research book. Our aim is to introduce the physics of solitons in a pedagogical way, so that the book can be read by a bachelor student or a student beginning master studies. We only assume a basic knowledge of physics, analytical mechanics and quantum mechanics. This book evolved from a course given by Michel Peyrard at the University of Dijon, and then at Ecole Normale Supérieure de Lyon for students graduating in ‘Statistical Physics and Nonlinear Science’. This course is now taught by Thierry Dauxois in the Master in physics and chemistry course at Ecole Normale Supérieure de Lyon.

The book does not claim to be exhaustive, however it is written in order to give a broad and fairly complete view of the topic. Part I contains the basics. It introduces the main classes of soliton equations from examples chosen in macroscopic physics. Theoretical methods are presented in Part II. Their selection has been made keeping in mind physical applications and this shows up in Parts III and IV which are devoted to selected topics in solid state or biomolecular physics. Modelling is a very important step, which starts from the physical system and leads to nonlinear equations which describe its properties. This aspect is illustrated in most of the chapters, but we also decided to devote a specific chapter to this question (Chapter 4) because this is a difficult and very important point.

Thinking in terms of solitons shines a new light on some physical systems. For instance, we show how solitons can be used to study the statistical physics of ferroelectric materials (Chapter 10) or of DNA (Chapter 16). We provide numerous bibliographical references which should allow the reader to go beyond the material presented in the book and find the elements to start research related to the physics of solitons. We also decided to include some biographical data on the scientists who founded this topic because we agree with the statement of the philosopher Whitehead who said, at the beginning of the twentieth century, ‘a science which does not want to remember its pioneers is condemned’.

This book matured after years of lectures and problem-solving classes, but we are also very grateful to those who devoted time to critical reading, particularly Geneviève Peyrard for her numerous pertinent remarks. Many colleagues, Mariette Barthès, Freddy Bouchet, Hervé Courtois, Jacques Dauxois, Sébastien Dusuel, Jean-Noël Gence, Hajime Hirooka, Robin Kaiser, Ioannis Kourakis, Juan Mazo, Guy Millot, Jean-Pierre Nguenang, Sébastien Paulin, Hicham Qasmi, Florence Raynal, Stefano Ruffo, Nobuhiko Saito, Yves-Henri Sanéjouand and Nikos Theodorakopoulos, examined chapters close to their research area. We thank also Larissa Brizhik, Lincoln D. Carr, Thierry Cretegny, Bernard Deconinck, Chris Eilbeck, Ying Li, Robert I. Odom, Sylvian R. Ray, Harvey Segur, Terry Toedtemeier, Nadezhda Tsypkina, Kathleen T. Zanotti for permission to use some pictures and Martin D. Kruskal and Norman J. Zabusky who provided us precise data on the early history of ‘solitons’.

A French edition of this book has been published under the title ‘*Physique des Solitons*’ by EDP Sciences/CNRS Edition in 2004. The English edition is a translation of the French edition, which has been slightly revised and completed. Chapter 14 has been added.