1

Continued fractions: real numbers

1.1 Historical background

1 Euclidean algorithm. Any pair $x_0 > x_1$ of positive integers generates a decreasing sequence $x_0 > x_1 > x_2 > \cdots$ in the set $\mathbb{N}$ of all positive integers:

$$\begin{aligned} x_0 &= b_0 x_1 + x_2, \\ x_1 &= b_1 x_2 + x_3, \\ x_2 &= b_2 x_3 + x_4, \\ &\vdots \\ x_{n-2} &= b_{n-2} x_{n-1} + x_n, \\ x_{n-1} &= b_{n-1} x_n, \end{aligned} \tag{1.1}$$

with $b_j \in \mathbb{N}$, $j = 0, 1, \ldots$ Since any decreasing sequence in $\mathbb{N}$ is finite, there exists $n \in \mathbb{N}$ such that for $x_{n-1} = b_{n-1} x_n$ the algorithm stops at this line.

Reading the equations in (1.1) from the top to $x_{n-2} = b_{n-2} x_{n-1} + x_n$, which precedes the last equation $x_{n-1} = b_{n-1} x_n$, we obtain that any common divisor of x_0 and x_1 divides x_n. Reading the same equations from the bottom to the top, we obtain that x_n is a common divisor of x_0 and x_1. Hence x_n is the greatest common divisor $d = (x_0, x_1)$ for x_0 and x_1. This is the standard form of the Euclidean algorithm, which provides a foundation for multiplicative number theory.

To explain the role played by the coefficients b_k in (1.1) we will consider (1.1) as a system of linear algebraic equations with integer coefficients $b_0, b_1, b_2, \ldots$ Eliminating the unknowns x_k from (1.1) we obtain

$$\frac{x_{k-1}}{x_k} = b_{k-1} + \frac{1}{x_k / x_{k+1}}, \qquad k = 1, 2, \ldots,$$

which obviously yields the development of x_0/x_1 into a finite *regular continued fraction*

$$\frac{x_0}{x_1} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots + \cfrac{1}{b_n}}}$$

To save space, Rogers (1907) proposed that the following notation could be used, in which the continued fraction is written in line form:

$$\frac{x_0}{x_1} = b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_{n-1}}. \tag{1.2}$$

This shows that any rational number equals the value of a regular continued fraction (1.2), where b_0 is an integer ($b_0 \in \mathbb{Z}$) and $b_1, b_2, \ldots, b_{n-1}$ are positive integers. The advantage of such a representation compared with popular decimal or dyadic representations is that it is universal and does not reflect particular properties of the base. *Thus the continuum* $\mathbb{R}$ *of real numbers can be parameterized by a sequence of integer parameters* $\{b_k\}_{k\geqslant 0}$ restricted to $b_0 \in \mathbb{Z}$ and $b_k \in \mathbb{N}$ if $k \geqslant 1$.

2 Hippasus of Metapontum. The algebraic construction of continued fractions discussed above originates in one important problem of geometry solved by the Pythagorean Hippasus of Metapontum in the fifth century BC. By the way, this problem is related to the notion of orthogonality; namely, given $AB \perp AD$, $x_1 = |AB| = |AD|$, prove that BD, $|BD| = x_0$ and AD have no common unit of measurement.

Hippasus' geometrical construction is remarkably similar to the construction of continued fractions (see Fig. 1.1). First, $x_0 > x_1 > x_2 = |ED|$, where E is defined so

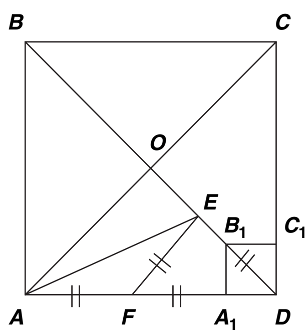

Fig. 1.1. Hippasus' construction for $x_1 = 2x_2 + x_3$.

that $|AB| = |BE|$. Computations with the angles in ΔABE, ΔAEF and ΔFED show that $|AF| = |FE| = |ED|$. Hence

$$x_0 = x_1 + x_2;$$
$$x_1 = 2x_2 + x_3, \quad |A_1D| = x_3 < x_2.$$

Observing that $\Delta ABD \sim \Delta EFD$, we have $x_2 = 2x_3 + x_4$. The construction can now be run by induction and it will never stop (notice that A_n never equals D). The result is that x_0/x_1 can be represented by an infinite continued fraction:

$$\frac{x_0}{x_1} = 1 + \frac{1}{2} \, {}_{+} \, \frac{1}{2} \, {}_{+} \, \frac{1}{2} \, {}_{+} \, \frac{1}{2} \, {}_{+} \cdots . \tag{1.3}$$

Since rational numbers are values of finite regular continued fractions and the development into a regular continued fraction is unique, this, by the way, shows that $\sqrt{2} = |BD|/|AD|$ is an irrational number.

3 Bombelli's method. In *L'algebra* R. Bombelli (1572) considered a method of computation of square roots $\sqrt{N}$, where N is a positive integer which is not a perfect square. Let a be the greatest positive integer satisfying $a^2 < N$. Then $N = a^2 + r$ with $r > 0$ and

$$\sqrt{a^2 + r} = a + x \Leftrightarrow x = \frac{r}{2a + x},$$

implying that

$$\sqrt{N} = a + \frac{r}{2a} \, {}_{+} \, \frac{r}{2a} \, {}_{+} \, \frac{r}{2a} \, {}_{+} \cdots .$$

In particular for $N = 13$ we obtain

$$\sqrt{13} = 3 + \frac{4}{6} \, {}_{+} \, \frac{4}{6} \, {}_{+} \, \frac{4}{6} \, {}_{+} \cdots = 3 + \frac{2}{3} \, {}_{+} \, \frac{1}{3} \, {}_{+} \, \frac{1}{3} \, {}_{+} \, \frac{1}{3} \, {}_{+} \cdots .$$

4 Ascending continued fractions. It is easy to see that any finite regular continued fraction represents a rational number. To find the rational number corresponding to the continued fraction

$$2 + \frac{1}{3} \, {}_{+} \, \frac{1}{2} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{4} \, {}_{+} \, \frac{1}{2} \, {}_{+} \, \frac{1}{3}$$

we rewrite it, starting from the right-hand side of the above expression, in the form of an *ascending continued fraction*

$$\cfrac{1}{\cfrac{1}{\cfrac{1}{3} + 2} + 4} + \cdots ,$$

which in six steps of elementary arithmetic operations results in 825/359.

5 Huygens' method. The theory of regular continued fractions originates in the practical problem of the approximation in the lowest terms of rational numbers with large numerators and denominators by rational numbers with much smaller ones. The first such problem was considered systematically by Huygens (1698). In this book Huygens studied a planetarium problem. For a planetarium to work accurately one should arrange the gear ratio to be

$$\frac{77\,708\,431}{2\,640\,858}.$$

Since it was impossible to arrange this ratio in practice, Huygens developed the ratio into the continued fraction

$$29+\frac{1}{2}\,\frac{1}{+\,2}\,\frac{1}{+\,1}\,\frac{1}{+\,5}\,\frac{1}{+\,1}\,\frac{1}{+\,4}\,\frac{1}{+\,1}\,\frac{1}{+\,1}\,\frac{1}{+\,2}\,\frac{1}{+\,1}\,\frac{1}{+\,6}\,\frac{1}{+\,1}\,\frac{1}{+\,10}\,\frac{1}{+\,2}\,\frac{1}{+\,2}\,\frac{1}{+\,3},$$

and studied the successive approximations

$$\begin{aligned}
\frac{P_{-1}}{Q_{-1}} &= \frac{1}{0} = +\infty,\\
\frac{P_0}{Q_0} &= \frac{29}{1} = 29,\\
\frac{P_1}{Q_1} &= 29+\frac{1}{2} = \frac{59}{2} = 29.5,\\
\frac{P_2}{Q_2} &= 29+\frac{1}{2}\,\frac{1}{+\,2} = \frac{147}{5} = 29.4,\\
\frac{P_3}{Q_3} &= 29+\frac{1}{2}\,\frac{1}{+\,2}\,\frac{1}{+\,1} = \frac{206}{7} = 29.428\,571\,43\ldots,\\
\frac{P_4}{Q_4} &= 29+\frac{1}{2}\,\frac{1}{+\,2}\,\frac{1}{+\,1}\,\frac{1}{+\,5} = \frac{1177}{40} = 29.425,\\
\frac{P_5}{Q_5} &= 29+\frac{1}{2}\,\frac{1}{+\,2}\,\frac{1}{+\,1}\,\frac{1}{+\,5}\,\frac{1}{+\,1} = \frac{1383}{47} = 29.425\,531\,91\ldots,\\
\frac{P_6}{Q_6} &= 29+\frac{1}{2}\,\frac{1}{+\,2}\,\frac{1}{+\,1}\,\frac{1}{+\,5}\,\frac{1}{+\,1}\,\frac{1}{+\,4} = \frac{6709}{228} = 29.425\,438\,60\ldots
\end{aligned} \tag{1.4}$$

A simple analysis of (1.2) shows that the value of the continued fraction lies between its consecutive *convergents* P_k/Q_k. Therefore to estimate the approximation error one can simply find the following differences:

$$\frac{59}{2}-\frac{29}{1}=\frac{1}{2\times 1},\ \frac{59}{2}-\frac{147}{5}=\frac{1}{2\times 5},$$
$$\frac{206}{7}-\frac{147}{5}=\frac{1}{7\times 147},\ \frac{206}{7}-\frac{1177}{40}=\frac{1}{7\times 40},\tag{1.5}$$
$$\frac{1383}{47}-\frac{1177}{40}=\frac{1}{47\times 40},\ \frac{1383}{47}-\frac{6709}{228}=\frac{1}{47\times 228}=\frac{1}{10\,716}.$$

The fact that all these differences are *aliquot* fractions, i.e. fractions with unit numerators and integer denominators, cannot be accidental. Basically it is this fact which yields a good rational approximation, 6709/228, to Huygens' fraction.

6 Continued fractions and the Gregorian calendar. Following Euler (1748), we consider an application of continued fractions to the calendar problem.

Problem 1.1 *Precise astronomical observations show that one year lasts*

$$365^d 5^h 48^m 55^s.$$

Find a calendar that will not accumulate a noticeable error for a long interval of time.

The assumption that one year lasts 365 days leads to an error of 5 hours per year. The error accumulates fairly fast and in 100 years results in a noticeable shift of the seasons. If we assume that one year lasts 366 days the disagreement with the seasons will be observed much earlier.

To solve this problem we first express the duration of one year in days:

$$1\text{ year}=365+\frac{5}{24}+\frac{48}{60}\times\frac{1}{24}+\frac{55}{60}\times\frac{1}{60}\times\frac{1}{24}\text{ days}=365+\frac{20\,935}{86\,400}\text{ days}.$$

This is itself, of course, an approximate duration but the error is so small that it will not be noticeable for more than 10 000 years.

To find a good approximation to 20 935/86 400 we develop this rational number into a regular continued fraction. It is clear that the numbers 20 935 and 86 400 are both divisible by 5, so that

$$\frac{20\,935}{86\,400}=\frac{4187}{17\,280}.$$

One can easily prove that the last fraction is in the lowest terms. Indeed $20\,935=5\times 53\times 79$, whereas $86400=2^7\times 3^3\times 5^2$, which implies that 5 is the greatest common divisor.

We have

$$\frac{4187}{17\,280} = \frac{1}{4} + \frac{532}{4187} = \frac{1}{4} + \frac{1}{7} + \frac{463}{532} = \frac{1}{4} + \frac{1}{7} + \frac{1}{1} + \frac{69}{463}$$
$$= \frac{1}{4} + \frac{1}{7} + \frac{1}{1} + \frac{1}{6} + \frac{49}{69} = \frac{1}{4} + \frac{1}{7} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \frac{20}{49}$$
$$= \frac{1}{4} + \frac{1}{7} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{2}{9}$$
$$= \frac{1}{4} + \frac{1}{7} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{2}.$$

The convergents to this continued fraction can be arranged into the following table:

$$\begin{array}{ccccccccc} 0 & 4 & 7 & 1 & 6 & 1 & 2 & 2 & \cdots \\ \frac{1}{0} & \frac{0}{1}^{l} & \frac{1}{4}^{g} & \frac{7}{29}^{l} & \frac{8}{33}^{g} & \frac{55}{227}^{l} & \frac{63}{260}^{g} & \frac{181}{747}^{l} & \cdots \end{array} \tag{1.6}$$

The first row of this table contains the partial denominators of the continued fractions. The second row consists of the corresponding convergents, shifted to the right by 1. This convenient notation is due to Euler and is explained below, in the paragraph before Theorem 1.5. The index l means that the convergent is less than the value of the continued fraction. The index g means that the convergent is greater than this value. It follows that every four years contribute a little bit less than 1 extra day. This gives rise to the Julian Calendar, which adds one extra day (29 February) every leap year (i.e. each year that is divisible by 4).

It should also be clear from (1.6) that every 33 years contribute a little bit less than eight days. Since

$$100 = 3 \times 33 + 1,$$

we obtain that every $400 = 4 \times 3 \times 33 + 4$ years contribute a little bit less than $4 \times 3 \times 8 + 1 = 97$ extra days. To arrange a convenient compensation, the Gregorian Calendar converts $3 = 100 - 97$ leap years within the range of every 400 years into ordinary years. Thus 1700, 1800, 1900 were ordinary years (to remove the three extra days contributed by the Julian Calendar). However, 1600 and 2000 were leap years. Since

$$\frac{97}{400} - \frac{4187}{17\,280} = 0.000\,197\,456\,\ldots,$$

the Gregorian Calendar contributes about two extra days every 10 000 years.

The Gregorian Calendar was introduced in 1582 by Pope Gregory. By that time the difference between the two calendars was already 10 days. The new calendar was introduced on 5 October 1582 and, to compensate the difference of 10 days, the day of 5 October 1582 was announced to be 15 October 1582 (Kiselev 1915, §97). It is interesting to notice that although the Gregorian calendar, as shown above, is closely related to continued fractions it was one of the contributors to the field at this time, Wallis, who advised the British authorities to reject it in Great Britain (Zeuthen 1903).

7 The well-tempered clavier. Here is an impressive application of continued fractions to music. The Weber–Fechner law states that the response of human beings to physical phenomena obeys a logarithmic law (see Maor 1994, pp. 111–12). This ability of human beings makes them less sensitive to the changes of the outside world by converting outside impulses with exponential growth into a linear response scale and so reduces our reaction to the most significant ones. In particular, our ear registers not the direct frequency ratio of two sounds but its logarithm. The main problem in music is to arrange a system of sounds which will create an impression of harmony under this logarithmic law of response. In practice this means that the frequencies in a musical scale should correspond to a linear set of logarithmic responses, i.e. these responses should divide up the logarithmic image of the scale into a number of equal parts. If a string of length l creates a sound of frequency $\omega = 512$ Hz then a string of length $l/2$ doubles the frequency to 2ω. The logarithmic base a is then chosen so as to normalize the following number to unity:

$$\log_a (2\omega : \omega) = \log_a 2 = 1,$$

which implies that $a = 2$. The ratio $2\omega : \omega = 2$ determines an interval $(\omega, 2\omega)$, called an *octave*. The ratio $3\omega/2 : \omega$ corresponding to half the interval $(\omega, 2\omega)$ (the frequency $3\omega/2$ is generated by a string of length $2l/3$) is called a *perfect fifth*; the ear hears this ratio as

$$\log_2 \left(\tfrac{3}{2}\omega : \omega\right) = \log_2 3 - 1.$$

Our ear hears a perfect fifth best, and therefore one should divide the logarithmic image of an octave into a number of equal parts in such a way that the above logarithmic image of a perfect fifth is well approximated. It can be shown that

$$\log_2 3 - 1 = 0.584\,962\,500\,721\ldots = \frac{1}{1}\,{+}\,\frac{1}{1}\,{+}\,\frac{1}{2}\,{+}\,\frac{1}{2}\,{+}\,\frac{1}{3}\,{+}\,\frac{1}{1}\,{+}\,\frac{1}{5}\,{+}\,\frac{1}{2}\,{+}\,\frac{1}{23}\,{+}\cdots$$

The convergents to the continued fraction of $\log_2 3 - 1$ form the series

$$1, \quad \frac{1}{2}, \quad \frac{3}{5}, \quad \frac{7}{12}, \quad \frac{24}{41}, \quad \ldots \tag{1.7}$$

and represent successive ways of dividing up the image of the octave. The approximations 1 and 1/2 are too inexact. The approximation 3/5 is used in eastern music. The approximation 7/12 is the best. It divides the octave into 12 semitones and seven such semitones correspond to a fifth. To study what happens at other frequencies we observe that the distance between two notes measured as the ratio of their frequencies is called an interval. If the interval between two notes is a ratio of small integers these two notes are called consonant. Otherwise they are called dissonant.[1] There are seven

[1] An original theory of sound classification by the "degree of pleasure" was developed by Euler in his monograph (1739).

intervals that are commonly considered as consonant (they had appeared already in Descartes' table; see Brouncker 1653, p. 13):

2/1 (octave)	5/4 (major third)
3/2 (perfect fifth)	6/5 (minor third)
4/3 (perfect fourth)	5/3 (major sixth)
	8/5 (minor sixth)

The analysis of these numbers shows that they form the sequence

$$1 < \frac{6}{5} < \frac{5}{4} < \frac{4}{3} < \frac{3}{2} < \frac{8}{5} < \frac{5}{3} < 2 \tag{1.8}$$

satisfying the relations

$$\frac{5}{3} \times \frac{6}{5} = \frac{5}{4} \times \frac{8}{5} = \frac{3}{2} \times \frac{4}{3} = 2, \qquad \frac{5}{4} \times \frac{6}{5} = \frac{3}{2}.$$

This implies that the binary logarithms of these intervals are linear combinations of 1, $\log_2 3/2$ and $\log_2 5/4$ with coefficients in $\{0, 1, -1\}$. Hence the error in the approximation by a uniform scale is completely determined by the errors for $\log_2 3/2$ and $\log_2 5/4$ and cannot exceed the maximum of the two. Now

$$\log_2\left(\frac{5}{4}\right) = 0.321\,928\,094\,887\,362\,347\,87\ldots = \frac{1}{3}\,{}_{+}\,\frac{1}{9}\,{}_{+}\,\frac{1}{2}\,{}_{+}\,\frac{1}{2}\,{}_{+}\,\frac{1}{4}\,{}_{+}\,\frac{1}{6}\,{}_{+}\cdots$$

shows that $1/3 = 4/12$ is a convergent to $\log_2 5/4$. This guarantees that the equal-temperament system of 12 uniform semitones gives a good rational approximation to the two basic intervals 3/2 and 5/4, and hence to all seven consonant intervals.

See Dunne and McConnell (1999) for a more detailed discussion. Excellent comments on this topic can also be found in the appendix by V. G. Boltyanskii to the Russian translation of Klein (1932).

8 Quadrature of the unit circle. The unit circle $\mathbb{T}$ is the boundary of the unit disc $\mathbb{D}$ centered at zero. The area of $\mathbb{D}$ is denoted by π. Having been introduced by W. Jones in (1706), the notation π became standard only after Euler published his monograph (1748). According to Cajori (1916, p. 32), in introducing π Jones was probably motivated by Oughtred's notation π/δ for the ratio of the circumference and the diameter of a circle; Oughtred was Wallis' teacher.

Theorem 1.2 (Archimedes) *The length of* $\mathbb{T}$ *is* 2π.

Proof Let us inscribe a regular n-polygon P_n in $\mathbb{T}$. Then its sides of equal length l_n make small triangles with the origin of area $l_n(1-o(1))/2$. Hence the area of P_n is $n \times l_n(1-o(1))/2$. It approaches π as $n \to +\infty$, whereas nl_n approaches the length of $\mathbb{T}$, which proves the theorem. □

Problem 1.3 (The quadrature problem) *Find a good rational approximation to the length of* $\mathbb{T}$.

The quadrature (squaring) of the circle was one of the most difficult ancient mathematical problems. Attempts to solve it resulted in significant progress in mathematical analysis and especially in the theory of continued fractions. The meaning of the problem has been changing in mathematics with time. In Archimedes' time the practical side of the problem was to construct with ruler and compass the side of a square having area π. The theoretical side of the problem was to either prove that $\pi \in \mathbb{Q}$ or at least find a good rational approximation to π. Now with Wolfram's Mathematica program[2] everybody can find thousands of digits of π:

$$\pi = 3.141\,592\,653\,589\,793\,238\,462\,643\,383\,279\ldots,$$

but originally the calculation of the correct decimal places of π was a difficult problem. The first important contribution to the quadrature problem was made by Archimedes, who developed the method of inscribed and superscribed regular n-polygons with $n = 6,\ 12,\ 24,\ 48,\ 96, \ldots, 3 \times 2^k$. For instance, consideration of a regular hexagon inscribed in a circle shows that $3 < \pi$.

Archimedes method looks especially beautiful in the form of Gregory (1667); see O'Connor and Robertson (2004). Let a_k be the semiperimeter of a regular n-polygon ($n = 3 \times 2^k$) inscribed in $\mathbb{T}$. In Fig. 1.2 its side is AC ($k = 1$), $AP = AC/2$ and AB is half the side of a superscribed regular n-polygon with semiperimeter b_k. It follows that $\angle AOB = \pi/n$. Since ΔOAB is similar to ΔOAP, we obtain that $|AB| = \tan \pi/n$ and $|AP| = \sin \pi/n$. Hence

$$a_k = n \sin \frac{\pi}{n} < \pi < b_k = n \tan \frac{\pi}{n}.$$

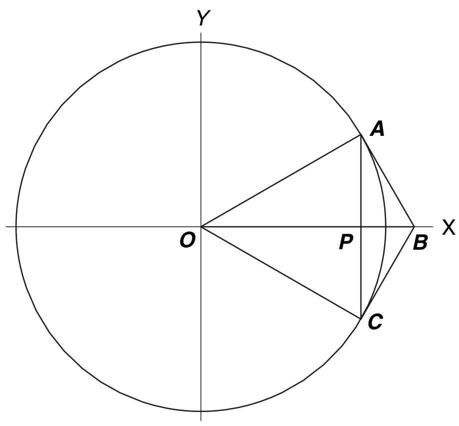

Fig. 1.2. Archimedes' construction to find an approximation to π.

[2] www.wolfram.research.com.

Clearly $a_1 = 3$, $b_1 = 2\sqrt{3} = 3.464101615\ldots$ Obvious trigonometry,

$$\frac{1}{\tan\theta} + \frac{1}{\sin\theta} = \frac{1}{\tan(\theta/2)}, \qquad 2\tan\frac{\theta}{2}\sin\theta = \left(2\sin\frac{\theta}{2}\right)^2,$$

results in the recurrence relations

$$b_{n+1} = \frac{2a_n b_n}{a_n + b_n}, \qquad a_{n+1} = \sqrt{a_n b_{n+1}}.$$

Archimedes obtained his inequalities $3.1410 < \pi < 3.1427$ by calculating a_5 and b_5. His method was considerably improved by Huygens (see Rudio 1892).

The first algebraic algorithm for the calculation of an arbitrary number of places of π was proposed by Brouncker in 1656;[3] see §**63** in Section 3.2. Although Brouncker's simple calculations remained unnoticed,[4] the more complicated calculations of Huygens resulted in significant progress in the rational approximation of π. In particular, it was shown that

$$\pi = 3 + \frac{1}{7} \, {}_+ \, \frac{1}{15} \, {}_+ \, \frac{1}{1} \, {}_+ \, \frac{1}{292} \, {}_+ \, \frac{1}{1} \, {}_+ \, \frac{1}{1} \, {}_+ \, \frac{1}{1} \, {}_+ \, \frac{1}{2} \, {}_+ \, \frac{1}{1} \, {}_+ \, \frac{1}{3} \, {}_+ \, \frac{1}{1} \, {}_+ \, \frac{1}{14} \, {}_+ \, \frac{1}{2} \, {}_+ \, \frac{1}{1} \, {}_+ \cdots.$$

In a similar way to (1.6), the convergents to the continued fraction of π can be arranged in the following table:

$$\begin{array}{ccccccc} 3 & 7 & 15 & 1 & 292 & 1 & \ldots \\ \frac{1}{0}^{g} & \frac{3}{1}^{l} & \frac{22}{7}^{g} & \frac{333}{106}^{l} & \frac{355}{113}^{g} & \frac{103\,993}{33\,102}^{l} & \ldots \end{array} \tag{1.9}$$

The approximation 355/113 is called Metius' approximation. The error in Metius' approximation is less than

$$\begin{aligned} 0 < \frac{355}{113} - \pi < \frac{355}{113} - \frac{103\,993}{33\,102} &= \frac{11\,751\,210 - 11\,751\,209}{113 \times 33\,102} \\ &= \frac{1}{113 \times 33\,102} = \frac{1}{3\,740\,526} \\ &= 0.000\,000\,267\ldots \end{aligned}$$

One can easily check with Wolfram's Mathematica program that there is another dramatic jump in the series of moderately small values of partial denominators for π. It happens at $n = 431$: $b_{431} = 20\,776$, whereas $b_{430} = 4$ and $b_{432} = 1$.

By Corollary 1.16 the continued fraction for π is finite if and only if $\pi \in \mathbb{Q}$. If π were a rational number then the quadrature of the circle would have a positive solution. Indeed, using ruler and compass one can easily construct any rational number a on the number axis. Then $\sqrt{a}$ is the length of the diagonal of the square with side a. Using a continued fraction for the cotangent of an angle, discovered by Euler, Lambert in (1761) proved that $\pi \notin \mathbb{Q}$. Later Legendre gave a simpler proof. We discuss this in more detail later in §**113** at the start of Section 5.3. The quadrature problem in the most general form was solved in 1882 by Lindemann, who showed that π does not satisfy any algebraic equation with integer coefficients; for

[3] According to a letter from Wallis to Digby sent on 6 June 1657.
[4] See §**63**, noting that Huygens was informed about Brouncker's calculations at his request.