RIEMANNIAN GEOMETRY

A Modern Introduction

Second Edition

This book provides an introduction to Riemannian geometry, the geometry of curved spaces, for use in a graduate course. Requiring only an understanding of differentiable manifolds, the book covers the introductory ideas of Riemannian geometry, followed by a selection of more specialized topics. Also featured are Notes and Exercises for each chapter to develop and enrich the reader's appreciation of the subject. This second edition has a clearer treatment of many topics from the first edition, with new proofs of some theorems. Also a new chapter on the Riemannian geometry of surfaces has been added.

The main themes here are the effect of curvature on the usual notions of classical Euclidean geometry, and the new notions and ideas motivated by curvature itself. Among the classical topics shown in a new setting is isoperimetric inequalities – the interplay of volume of sets and the areas of their boundaries – in curved space. Completely new themes created by curvature include the classical Rauch comparison theorem and its consequences in geometry and topology, and the interaction of microscopic behavior of the geometry with the macroscopic structure of the space.

Isaac Chavel is Professor of Mathematics at The City College of the City University of New York. He received his Ph.D. in Mathematics from Yeshiva University under the direction of Professor Harry E. Rauch. He has published in international journals in the areas of differential geometry and partial differential equations, especially the Laplace and heat operators on Riemannian manifolds. His other books include *Eigenvalues in Riemannian Geometry* (1984) and *Isoperimetric Inequalities: Differential Geometric and Analytic Perspectives* (2001). He has been teaching at The City College of the City University of New York since 1970, and he has been a member of the doctoral program of the City University of New York since 1976. He is a member of the American Mathematical Society.

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for
HARRY ERNEST RAUCH
(1925–1979)

Contents

Preface to the Second Edition

In this second edition, the first order of business has been to correct mistakes, mathematical and typographical, large and small, and clarify a number of arguments that were unclear or given short shrift the first time round. I can only hope that, in this process, and in the process of changes and additions described below, I have not introduced any new errors.

I have added some proofs of theorems, and sketches to some of the exercises, that were originally left completely to the reader in the first edition. I have added some new notes and exercises as well.

In the text itself, I have made a few changes. I added a chapter with topics from surfaces, immediately following the chapter on coverings (Chapter IV). The chapter (Chapter V) now includes the Gauss–Bonnet theorem; but, it also contains topics of current interest, showing that the Riemannian geometry of surfaces is alive and well, and is a constant testing ground, as well as a source, of new ideas. As it contained the introduction to the isoperimetric problem in Riemannian manifolds, presenting the Bol–Fiala inequalities, and the Benjamini–Cao solution of the isoperimetric problem on the paraboloid of revolution, I thought it best to follow the chapter with isoperimetric inequalities in the classical constant curvature space forms (Chapter VI).

This last chapter (Chapter VI) is a bit different from what I presented in the first edition. New proofs were given for the isoperimetric problem in Euclidean space, with the famous proof by M. Gromov, using Stokes' theorem, now appearing in my other book *Isoperimetric Inequalities: Differential Geometric and Analytic Perspectives* (2001). The Brunn–Minkowski inequalities in hyperbolic space and the sphere were redone, hopefully improving on the first presentation.

Chapter VI is followed by the original (now Chapter VII) on the kinematic density, with little change. I was sorely tempted to include the Burago–Ivanov solution to the E. Hopf conjecture that metrics on the torus, of *all* dimensions, without conjugate points are flat. But, such an undertaking would have taken the discussion too far afield. This chapter is then followed by the one on isoperimetric inequalities in general Riemannian manifolds, and the chapter on the Rauch comparison theorem and its consequences.

Beyond the Notes and Exercises sections that conclude each chapter, the reader is highly recommended to M. Berger's recent survey *A Panoramic View of Riemannian Geometry* (2003), preceded by his preparatory essay *Riemannian Geometry During the Second Half of the Twentieth Century* (2000). Just about every page of this introduction to Riemannian geometry could have contained references to Berger's surveys for further background and future work.

It is a pleasure to thank the readers of my first edition for their warm reception of the book and for the helpful criticisms – both in pointing out errors and in suggesting improvements. I should add that I found the reviews very helpful, and I am grateful for the effort that went into them. I hope this edition merits the effort they invested.

ISAAC CHAVEL
Riverdale, New York
February 2005

Preface

My goals in this book on Riemannian geometry are essentially the same as those that guided me in my *Eigenvalues in Riemannian Geometry* (1984): to introduce the subject, to coherently present a number of its basic techniques and results with a mind to future work, and to present some of the results that are attractive in their own right. This book differs from *Eigenvalues* in that it starts at a more basic level. Therefore, it must present a broader view of the ideas from which all the various directions emerge. At the same time, other treatments of Riemannian geometry are available at varying levels and interests, so I need not introduce everything. I have, therefore, attempted a viable introduction to Riemannian geometry for a very broad group of students, with emphases and developments in areas not covered by other books.

My treatment presupposes an introductory course on manifolds, the construction of associated tensor bundles, and Stokes' theorem. When necessary, I recall the facts and/or refer to the literature in which these matters are discussed in detail.

I have not hesitated to prove theorems more than once, with different points of view and arguments. Similarly, I often prove weaker versions of a result and then follow with the stronger version (instead of just subsuming the former result under the latter). The variety of levels, ideas, and approaches is a hallmark of mathematics; and an introductory treatment should display this variety as part of the development of broad technique and as part of the aesthetic appreciation of the mathematical endeavor.

I am confident that a short course could be easily crafted from Chapters I to IV and VII (the second edition: Chapter IX), and a more ambitious course from the remaining material. Every chapter of the book features a Notes and Exercises section. These sections cover (i) references to earlier literature and to other results; (ii) "toes in the water" introductions to topics emerging from the ideas presented in the main body of the text; and (iii) examples and applications. The Notes and Exercises sections of the first four chapters are quite extensive.

These sections in the later chapters are not as ambitious as those in the first four, since the first four chapters are genuinely introductory.

The Notes and Exercises sections are organized loosely under subheadings of topics. These are not to be taken too literally; rather, they attempt to restrain the variety of material in these sections from becoming chaotic.

A submotif in the Notes and Exercises sections is the method of calculation with moving frames, even though the method is not used extensively in the main body of the text itself. Besides the obvious claim that such calculations should be included in an introductory treatment, I had in mind a quiet tribute to the late William F. Pohl. I learned the magic long ago of *repère mobile* from Bill Pohl at the University of Minnesota. I can still see his full frame at the blackboard, extending his arm gracefully in front of him, moving his hands descriptively with his fingers playing the role of the frame vectors, and declaring that ${\omega_2}^3 = 0$ since the frame vector field e_2 did not turn in the direction of e_3.[1]

It is a pleasure to thank my colleagues and friends for their contributions to my work, in general, and to this book, in particular. P. Buser provided me with some helpful discussions and read portions of the work. So did J. Dodziuk and E. A. Feldman. Finally, I wish to thank the geometers of the doctoral program of the City University of New York, namely J. Dodziuk, L. Karp, B. Randol, R. Sacksteder, J. Velling, and Edgar A. Feldman – who have provided, over the years, all sorts of help, mathematical stimulation and insight, and scientific partnership. Their contribution permeates all the pages of this book.

ISAAC CHAVEL
Riverdale, New York
July 1992

[1] My memory hits the mark. As soon as I mentioned to Ed Feldman that I put moving frames in the book because of Bill Pohl, Ed performed an imitation of the grand gesture that was Bill's trademark.