Cambridge University Press
978-0-521-85337-8 — Free Ideal Rings and Localization in General Rings
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Excerpt
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Generalities on rings and modules

This chapter collects some facts on rings and modules, which form neither part of our subject proper, nor part of the general background (described in the Appendix). By its nature the content is rather mixed, and the reader may well wish to begin with Chapter 1 or even Chapter 2, and only turn back when necessary.

In Section 0.1 we describe the conditions usually imposed on the ranks of free modules. The formation of matrix rings is discussed in Section 0.2; Section 0.3 is devoted to projective modules and the special class of Hermite rings is considered in Section 0.4.

Section 0.5 deals with the relation between a module and its defining matrix, and in particular the condition for two matrices to define isomorphic modules. This and the results on eigenrings and centralizers in Section 0.6 are mainly used in Chapters 4 and 6.

The Ore construction of rings of fractions is behind much of the later development, even when this does not appear explicitly. In Section 0.7 we recall the details and apply it in Section 0.8 to modules over Ore domains; it turns out that the (left or right) Ore condition has some unexpected consequences. In Section 0.9 we recall some well-known facts on factorization in commutative rings, often stated in terms of monoids, in a form needed later.

0.1 Rank conditions on free modules

Let R be any ring, M an R-module and I a set. The direct power of M with index set I is denoted by M^I, while the direct sum is written $M^{(I)}$. When I is finite, with n elements, these two modules agree and are written as M^n, as usual. More precisely, M^n denotes the set of rows and ${}^nM$ the set of columns of length n.

With every left R-module M we can associate its dual

$$M^* = \mathrm{Hom}_R(M, {}_RR),$$

consisting of all linear functionals on M with the natural right R-module structure defined by $(\alpha c, x) = (\alpha, cx)$, where $x \in M$, $\alpha \in M^*$ and $c \in R$. Similarly, every right R-module N has as dual the left R-module $N^* = \mathrm{Hom}_R(N, R_R)$. In particular, $(R^n)^* \cong {}^nR$, $({}^nR)^* \cong R^n$; more generally, if P is a finitely generated projective left R-module, then P^* is a finitely generated projective right R-module and $P^{**} \cong P$. For if $P \oplus Q \cong R^n$, then $P^* \oplus Q^* \cong {}^nR$ and $P^{**} \oplus Q^{**} \cong R^n$. Now the obvious map $\delta_P : P \to P^{**}$, which maps $x \in P$ to $\hat{x} : f \mapsto \langle f, x\rangle$ is an isomorphism, because $\delta_P \oplus \delta_Q = 1$.

Let R be any ring and M a left R-module with a minimal generating set X. If X is infinite, then any generating set of M has at least $|X|$ elements, and in particular, any two minimal generating sets of M have the same cardinality. However, when X is finite, this need not be so, thus a free module on a finite free generating set may have minimal generating sets of different sizes. We shall say that R^n has *unique rank* if it is not isomorphic to R^m for any $m \neq n$. Using the pairing provided by * we see that R^n has unique rank if and only if ${}^nR$ has unique rank. For any free module F of unique rank n we write $n = \mathrm{rk}(F)$.

A ring R is said to have the *invariant basis property* or *invariant basis number* (IBN) if every free R-module has unique rank. Most rings commonly encountered have IBN, but we shall meet examples of non-zero rings where this property fails to hold.

Occasionally we shall need stronger properties than IBN. A ring R is said to have *unbounded generating number* (UGN) if for every n there is a finitely generated R-module that cannot be generated by n elements. Since any n-generator module is a homomorphic image of a free module of rank n, it follows that in a ring with UGN a free module of rank n cannot be generated by fewer than n elements, and this condition characterizes rings with UGN. It also shows that UGN implies IBN.

A ring R is said to be *weakly n-finite* if every generating set of n elements in R^n is free; if this holds for all n, R is called *weakly finite* (WF). Weakly 1-finite rings are sometimes called ‘directly finite’, ‘von Neumann finite’ or ‘inverse symmetric’. As an example of weakly finite rings we have projective-free rings, where a ring is called *projective-free* if every finitely generated projective module is free, of unique rank.

Let R be any non-zero ring and suppose that R^n has a generating set of m elements, for some $m, n \geq 1$. Then we have a surjection $R^m \to R^n$, giving rise to an exact sequence

$$0 \to K \to R^m \to R^n \to 0.$$

Since R^n is free, the sequence splits and so $R^m \cong R^n \oplus K$. This shows that the three properties defined here may be stated as follows:

IBN. *For all* m, n, $R^m \cong R^n$ *implies* $m = n$.
UGN. *For all* m, n, $R^m \cong R^n \oplus K$ *implies* $m \geq n$.
WF. *For all* n, $R^n \cong R^n \oplus K$ *implies* $K = 0$.

By describing the change of basis, we can express these conditions in matrix form:

IBN. *For any* $A \in {}^mR^n, B \in {}^nR^m$, *if* $AB = I_m$, $BA = I_n$, *then* $m = n$.
UGN. *For any* $A \in {}^mR^n, B \in {}^nR^m$, *if* $AB = I_m$, *then* $n \geq m$.
WF. *For any* $A, B \in R_n$, *if* $AB = I$, *then* $BA = I$.

We see that a ring has IBN if and only if every invertible matrix has index zero; it has UGN if and only if every matrix with a right inverse has non-negative index, and it is weakly finite if and only if all inverses of square matrices are two-sided. The UGN condition can also be defined in terms of the rank of a matrix, which over general rings is defined as follows. Given any matrix A, of the different ways of writing A as a product, $A = PQ$, we choose one for which the number of rows of Q is least. This number is called the *inner rank* of A, written $\rho(A)$ or ρA, and the corresponding factorization of A is called a *rank factorization*. For matrices over a field this notion of rank reduces to the familiar rank; now we observe that a ring has UGN if and only if the inner rank of any $n \times n$ unit matrix is n. Such a matrix is said to be *full*. Thus a matrix is full if and only if it is square, say $n \times n$, and cannot be written as a product of an $n \times r$ by an $r \times n$ matrix, where $r < n$. We note that every non-zero element (in any ring) is full as a 1×1 matrix, and the unit matrix of every size is full precisely if the ring has UGN. Over a field the full matrices are just the regular matrices (see Section 5.4), but in general there is no relation between full and regular matrices.

Either set of the above conditions makes it clear that the zero ring is weakly finite, but has neither IBN nor UGN. For a non-zero ring,

$$\text{WF} \Rightarrow \text{UGN} \Rightarrow \text{IBN},$$

and if a ring R has any of these properties, then so does its opposite R°. Moreover, if $R \to S$ is a homomorphism and S has IBN or UGN, then so does R. Clearly any field (even skew) has all properties; more generally this holds for any Noetherian ring (see BA, theorem 4.6.7 or Exercise 5 below), as well as any subring of a field. Using determinants, we see that every non-zero commutative ring also has all three properties. Examples of rings having IBN but not UGN, and rings having UGN but not weakly finite, may be found in Cohn [66a] or in SF, Section 5.7 (see also Exercise 2 and Section 2.11).

For a non-zero ring without IBN there exist positive integers h, k such that

$$R^h \cong R^{h+k}, \quad h, k \geq 1. \tag{1}$$

The first such pair (h, k) in the lexicographic ordering is called the *type* of the ring R. We observe that for a ring R of type (h, k) $R^m \cong R^n$ holds if and only if $m = n$ or $m, n \geq h$ and $m \equiv n (\mathrm{mod}\ k)$ (see, e.g. UA, Theorem X.3.2, p. 340).

Proposition 0.1.1. *Let $f: R \to S$ be a homomorphism between non-zero rings. If R does not have IBN and its type is (h, k), then S does not have IBN and if its type is (h', k'), then $h' \leq h, k'|k$.*

Here it is important to bear in mind that all our rings have a unit element, which is preserved by homomorphisms and inherited by subrings.

Proof. By hypothesis (1) holds, hence there exist $A \in {}^mR^n, B \in {}^nR^m$ satisfying $AB = \mathrm{I}, BA = \mathrm{I}$, with $m = h, n = h + k$. Applying f we get such matrices over S, whence it follows that $S^h \cong S^{h+k}$, so S cannot have IBN and $h' \leq h, k'|k$. ∎

The next result elucidates the connexion between weak finiteness and UGN.

Proposition 0.1.2. *A ring R has UGN if and only if some non-zero homomorphic image of R is weakly finite.*

Proof. If a non-zero homomorphic image S of R is weakly finite, then S has UGN, hence so does R. Conversely, assume that the zero ring is the only weakly finite homomorphic image of R. By adjoining the relations $YX = \mathrm{I}$, for all pairs of square matrices X, Y satisfying $XY = \mathrm{I}$, we obtain a weakly finite ring S. For suppose that $\mathrm{I} - AB = \Sigma_1^r U_i(\mathrm{I} - Y_i X_i)V_i$, where $X_i Y_i = \mathrm{I}$. By taking $X = X_1 \oplus \cdots \oplus X_r, Y = Y_1 \oplus \cdots \oplus Y_r, U = (U_1, \ldots, U_r), V = (V_1, \ldots, V_r)^{\mathrm{T}}$, we can write this as

$$\mathrm{I} - AB = U(\mathrm{I} - YX)V, \tag{2}$$

and $XY = \mathrm{I}$. If A, B are $n \times n$ and X, Y are $m \times m$, then U is $n \times m$ and V is $m \times n$. Suppose that $n \geq m$; on replacing X, Y by $X \oplus \mathrm{I}, Y \oplus \mathrm{I}$, respectively, where I is the unit matrix of order $n - m$, and completing U, V to square matrices by adding columns, respectively rows of zeros, we obtain an equation (2), where all matrices are square of order n. Similarly, if $n \leq m$, we can achieve the same result by taking diagonal sums of A, B with I. Writing $Z = AX + U(\mathrm{I} - YX), W = YB + (\mathrm{I} - YX)V$, we have $ZW = \mathrm{I}$ and $ZY = A, XW = B$, hence $\mathrm{I} - BA = X(\mathrm{I} - WZ)Y$. Therefore S is weakly finite and so must be the zero ring. It follows that R becomes zero by adjoining a finite number of such matrix equations, and by taking diagonal sums we obtain a single pair

X, Y, each $s \times s$, say, for which this happens. Thus $XY = \mathrm{I}$, while the ideal generated by the entries of $\mathrm{I} - YX$ is the whole ring. Replacing each of X, Y by a diagonal sum of an appropriate number of copies, we may assume that there exist $p \in R^s$, $q \in {}^sR$ such that $p(\mathrm{I} - YX)q = 1$. Therefore we have

$$\mathrm{I}_{s+1} = \begin{pmatrix} X \\ p(\mathrm{I} - YX) \end{pmatrix} \begin{pmatrix} Y & (\mathrm{I} - YX)q \end{pmatrix},$$

and this equation shows that UGN fails for R. ■

For another characterization of weak finiteness we shall need to refine the notion of inner rank. Given a non-zero ring R, let A be any matrix over R and consider $A \oplus \mathrm{I}_r$, the diagonal sum of A and the $r \times r$ unit matrix. Since any factorization of A can also be used to factorize $A \oplus \mathrm{I}_r$, it follows that $\rho(A \oplus \mathrm{I}_r) \leq \rho(A) + r$. Therefore we have

$$\rho(A) \geq \rho(A \oplus \mathrm{I}_1) - 1 \geq \rho(A \oplus \mathrm{I}_2) - 2 \geq \ldots \tag{3}$$

If this sequence has a finite limit, we denote it by $\rho^*(A)$ and call it the *stable rank* of A. An $n \times n$ matrix of stable rank n is said to be *stably full*. Thus a square matrix A is stably full if and only if $A \oplus \mathrm{I}_r$ is full for all $r \geq 1$. Hence every stably full matrix is full, but the converse need not hold.

For a ring without UGN the unit matrix of some size is not full, say the $n \times n$ unit matrix I_n has rank $n - 1$. Then the sequence (3) is unbounded below and we formally put $\rho^*(A) = -\infty$ for every matrix A. If R has UGN, we have $\rho(A \oplus \mathrm{I}_r) \geq r$, so in this case the sequence (3) is bounded below by 0 and hence has a limit; thus in any ring with UGN the stable rank of every matrix exists as a non-negative integer. Conversely, if the stable rank exists for some matrix A, say $\rho^*(A) = r$, then for some n and all $s \geq n$, $\rho(A \oplus \mathrm{I}_s) = r + s$. Hence for any $t \geq 0, r + s + t = \rho(A \oplus \mathrm{I}_s \oplus \mathrm{I}_t) \leq \rho(A \oplus \mathrm{I}_s) + \rho(\mathrm{I}_t) = r + s + \rho(\mathrm{I}_t)$. Thus $\rho(\mathrm{I}_t) \geq t$ and this proves that R has UGN. We now have the following connexion with weak finiteness.

Proposition 0.1.3. *For any non-zero ring R the following are equivalent:*

(a) R is weakly finite,
(b) every non-zero matrix over R has a stable rank, which is positive,
(c) every non-zero idempotent matrix over R has a stable rank, which is positive.

Proof. We note that in each case the stable rank is finite. Now let A be any $m \times n$ matrix over R, of stable rank t, say; then for some $s \geq 0$, $\rho(A \oplus \mathrm{I}_s) = t + s = r$, say. So we can write

$$\begin{pmatrix} A & 0 \\ 0 & \mathrm{I}_s \end{pmatrix} = \begin{pmatrix} B \\ B' \end{pmatrix} \begin{pmatrix} C & C' \end{pmatrix}, \tag{4}$$

where $B \in {}^mR^r, B' \in {}^sR^r, C \in {}^rR^n, C' \in {}^rR^s$. Thus we have $B'C' = \mathrm{I}, BC' = 0 = B'C, BC = A$.

To prove (a) ⇒ (b), assume that $\rho^*(A) = 0$; then $r = s$, so by weak finiteness, $C'B' = \mathrm{I}$, hence $B = 0 = C$ and therefore $A = BC = 0$.

(b) ⇒ (c) is clear; to prove (c) ⇒ (a), assume that R is not weakly finite. Then there exist $B', C' \in R_s$ with $B'C' = \mathrm{I}, C'B' \neq \mathrm{I}$, so (4) holds with $m = n = r = s, A = B = C = \mathrm{I} - C'B'$, and this is a non-zero idempotent matrix of zero stable rank. ∎

In conclusion we note another consequence of weak finiteness.

Proposition 0.1.4. *Let R be a weakly n-finite ring and let $A \in {}^rR^n, A' \in {}^nR^r, B \in {}^nR^s, B' \in {}^sR^n$ be such that $AB = 0, AA' = I_r, B'B = I_s$, where $r + s = n$. Then there exists $P \in GL_n(R)$ such that*

$$A = (\mathrm{I}_r \quad 0)P, \qquad B = P^{-1}\begin{pmatrix} 0 \\ \mathrm{I}_s \end{pmatrix}.$$

Proof. These equations just state that A constitutes the first r rows of P, while B forms the last s columns of P^{-1}. To prove this result, we have by hypothesis

$$\begin{pmatrix} A \\ B' \end{pmatrix}(A' \quad B) = \begin{pmatrix} \mathrm{I}_r & 0 \\ B'A' & \mathrm{I}_s \end{pmatrix},$$

where all the matrices are $n \times n$. By subtracting $B'A'$ times the first r rows from the last s we reduce the right-hand side to I, so the result follows by taking $P = (A, B'E)^{\mathrm{T}}, P^{-1} = (A', B)$, where $E = \mathrm{I} - A'A$. ∎

Exercises 0.1

1. Show that over a ring of type $(h, k)(k \geq 1)$ every finitely generated module can be generated by h elements. Find a bound for the least number of elements in a basis of a finitely generated free module.
2. If K is a non-zero ring and I an infinite set, show that $R = \mathrm{End}(K^{(I)})$ does not have IBN and determine its type.
3. If every finitely generated R-module is cyclic, show that R cannot be an integral domain; in particular, obtain this conclusion for a ring of type $(1, k)$.
4. A ring R is said to have *bounded decomposition type* (BDT), if there is a function $r(n)$ such that R^n can be written as a direct sum of at most $r(n)$ terms. Show that any ring with BDT is weakly finite.
5. Show that a ring with left ACC_n for some $n \geq 1$ is weakly n-finite. Deduce that a left (or right) Noetherian ring, or more generally, a ring with left (or right) pan-ACC is weakly finite. (Recall that 'pan-ACC' stands for 'ACC_n for all n'.) Obtain the same conclusion for DCC_n. (*Hint*: See Exercise 7.10.)

6. Let R be a non-zero ring without IBN and for fixed $m, n (m \neq n)$ consider pairs of mutually inverse matrices $A \in {}^mR^n$, $B \in {}^nR^m$. Show that if A', B' is another such pair, then $P = A'B$ is an invertible matrix such that $PA = A'$, $BP^{-1} = B'$. What is P^{-1}?
7. Let R be a weakly n-finite ring. Given maps $\alpha : R^r \to R^n$ and $\beta : R^n \to R^s (r + s = n)$ such that $\alpha\beta = 0$, α has a right inverse and β has a left inverse, then there exists an automorphism μ of R^n such that $\alpha\mu : R^r \to R^n$ is the natural inclusion and $\mu\pi = \beta$, where $\pi : R^n \to R^s$ is the natural projection. Show that conversely, every ring with this property is weakly n-finite. (*Hint*: Imitate the proof of Proposition 1.4.)
8. Show that a ring R is weakly n-finite if and only if (F): *Every surjective endomorphism of R^n is an automorphism.* If a non-zero ring R has the property (F), show that every free homomorphic image of R^n has rank at most n. Deduce that every non-zero weakly finite ring has UGN.

9*. Which of IBN, UGN, weak finiteness (if any) are Morita invariants?

10°. Characterize the rings all of whose homomorphic images are weakly finite.

11. (Leavitt [57]) Show that if a ring R has a non-zero free module F with no infinite linearly independent subset, then F has unique rank.

12*. (Montgomery [83]) Let A be an algebra over the real numbers with generators a_0, a_1, b_0, b_1 and defining relations $a_0b_0 - a_1b_1 = 1$, $a_1b_0 + a_0b_1 = 0$. Show (by using a normal form for the elements of A) that A is an integral domain, hence weakly 1-finite, but not weakly 2-finite. Show also that $A \otimes_R C$ is not weakly 1-finite (see also Exercise 2.11.8).

13°. Is the tensor product of two weakly finite k-algebras again weakly finite?

14°. Is every weakly 1-finite von Neumann regular ring weakly finite?

15. Let $V_{m,n}$ be a k-algebra with $2mn$ generators, arranged as an $m \times n$ matrix A and an $n \times m$ matrix B and defining relations (in matrix form) $AB = \mathrm{I}$, $BA = \mathrm{I}$ (the 'canonical non-IBN ring' for $m \neq n$). Show that $V_{1,n}$ is a simple ring for $n > 1$; what is $V_{1,1}$?

16. (M. Kirezci) If $V_{m,n}$ is defined as in Exercise 15 and $m < n$, show that there is a homomorphism $V_{m,n+r(n-m)} \to V_{m,n}$, for any $r > 0$. [*Hint*: If in $V_{m,n}$, $A = (A_1, A_2)$, $B = (B_1, B_2)^{\mathrm{T}}$, where A_1, B_1 are square, verify that $(A_1^r, A_1^{r-1}A_2, \ldots, A_2)$ and $(B_1^r, B_2B_1^{r-1}, \ldots, B_2)^{\mathrm{T}}$ are mutually inverse.] Deduce that $V_{1,n}$, for $n > 1$, can be embedded in $V_{1,2}$.

0.2 Matrix rings and the matrix reduction functor

Given a ring R, consider a left R-module which is expressed as a direct sum of certain submodules:

$$M = U_1 \oplus \cdots \oplus U_n. \tag{1}$$

Let $\pi_i : M \to U_i$ be the canonical projections and $\mu_i : U_i \to M$ the canonical injections, for $i = 1, \ldots, n$. Thus $(x_1, \ldots, x_n)\pi_i = x_i$, $x\mu_i = (0, \ldots, x, \ldots, 0)$ with x in the ith place and 0 elsewhere. Clearly we have

$$\mu_i\pi_j = \delta_{ij}, \tag{2}$$

where δ is the Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Further,

$$\sum \pi_i \mu_i = 1. \tag{3}$$

With each endomorphism f of M we can associate a matrix (f_{ij}), where $f_{ij}: U_i \to U_j$ is defined by

$$f_{ij} = \mu_i f \pi_j. \tag{4}$$

Similarly, any family of homomorphisms $f_{ij}: U_i \to U_j$ gives rise to an endomorphism f of M defined by

$$f = \sum \pi_i f_{ij} \mu_j. \tag{5}$$

These two processes are easily seen to be mutually inverse and if we add and multiply two families (f_{ij}) and (g_{ij}) 'matrix fashion': $(f + g)_{ij} = f_{ij} + g_{ij}$, $(fg)_{ik} = \Sigma f_{ij} g_{jk}$, the correspondence is an isomorphism, so that we have

Theorem 0.2.1. *Let R be any ring. If M is a left R-module, expressed as a direct sum as in (1), then each element f of $\mathrm{End}_R(M)$ can be written as a matrix (f_{ij}) where $f_{ij}: U_i \to U_j$, is obtained by (4) and in turn gives rise to an endomorphism of M by (5), and this correspondence is an isomorphism.* ■

In the particular case where all summands are isomorphic, we have $M \cong U^n$ and so we find

Corollary 0.2.2. *Let A be a ring, U a left A-module and $R = \mathrm{End}_A(U)$. Then for any $n \geq 1$ we have*

$$\mathrm{End}_A(U^n) \cong \mathfrak{M}_n(R). \quad ■ \tag{6}$$

The matrix ring $\mathfrak{M}_n(R)$ in (6) is also denoted by R_n. Let us consider it more closely. Writing $e_{ij} = \pi_i \mu_j$, we obtain from (2) and (3) the equations

$$e_{ij} e_{kl} = \delta_{jk} e_{il}, \quad \Sigma e_{ii} = 1. \tag{7}$$

The e_{ij} are just the matrix units and the matrix ring R_n may be defined as the ring generated by R and n^2 elements $e_{ij} (i, j = 1, \ldots, n)$ satisfying the conditions (7) and $ae_{ij} = e_{ij} a$ for all $a \in R$. The general element of R_n is then uniquely expressible as $\sum a_{ij} e_{ij}$ $(a_{ij} \in R)$. In fact matrix rings are characterized by (7), which gives a decomposition of 1 in R into n idempotents: $1 = e_{11} + \cdots + e_{nn}$.

Theorem 0.2.3. *Let S be any ring with n^2 elements e_{ij} satisfying the equations (7). Then $S \cong \mathfrak{M}_n(R)$, where R is the centralizer of all the e_{ij}.*

Proof. For each $a \in S$ we define $a_{ij} = \sum_\nu e_{\nu i} a e_{j\nu}$; then it is easily checked that $a_{ij} \in R$ and $a = \sum_{ij} a_{ij} e_{ij}$. Now the correspondence $a \leftrightarrow (a_{ij})$ is seen to be an isomorphism: $S \cong R_n$. ■

Using the language of categories, we can say that the process of forming the $n \times n$ matrix ring is a functor from Rg, the category of rings, to Rg_n, the category of $n \times n$ matrix rings: to each ring R corresponds the matrix ring R_n and to each ring homomorphism $f : R \to S$ there corresponds the homomorphism from R_n to S_n obtained by applying f to the separate matrix entries; conversely, any homomorphism $R_n \to S_n$ arises in this way from a homomorphism $R \to S$, because R is characterized within R_n as the centralizer of the e_{ij}. Moreover, every object T in Rg_n is of the form $\mathfrak{M}_n(C)$, where C is the centralizer of the e_{ij} in T. This shows the functor $\mathfrak{M}_n$ to be a category equivalence (BA, Proposition 3.3.1 or Appendix B below). Thus we have proved

Theorem 0.2.4. *The matrix functor $\mathfrak{M}_n$ establishes an equivalence between the categories Rg and Rg_n, for any $n \geq 1$.* ■

Of course this is just an instance of the well-known Morita equivalence (see Appendix B). Given a left A-module U with endomorphism ring $R = \text{End}_A(U)$, when we considered U^n as an A-module, its endomorphism ring turned out to be R_n. But we can also consider U^n as an A_n-module; in that case its endomorphism ring, i.e. the centralizer of A_n in $\text{End}(U^n)$, is the centralizer of the matrix basis $\{e_{ij}\}$ in R_n, i.e. R itself. Thus we have

$$\text{End}_{A_u}(U^n) \cong R. \tag{8}$$

In the two cases (6) and (8), U^n may be visualized as consisting of row vectors and column vectors, respectively, over U. We shall distinguish these cases by writing the set of column vectors as ${}^nU$ and the set of row vectors as U^n. More generally, we denote by ${}^mU^n$ the set of all $m \times n$ matrices with entries in U, and omit reference to either of m or n equal to 1. For a ring R, R_n is just ${}^nR^n$, considered as a ring. We shall also allow m or n to be 0. Thus ${}^0U^n$ is the set of matrices with no rows and n columns; there is one such matrix for each n (including $n = 0$). Similarly for ${}^mU^0$; of course R_0 is the zero ring. The unique 0×0 matrix over R will be written 1_0, and an $m \times n$ matrix where $mn = 0$ will be called a *null matrix*.

If M is an (R, S)-bimodule, then ${}^mM^n$ is an (R_m, S_n)-bimodule in a natural way. As an example, take R itself, considered as an R-bimodule; the set of row vectors R^n has a natural (R, R_n)-bimodule structure and the set of column vectors ${}^nR$ a natural (R_n, R)-bimodule structure. Writing R_R, ${}_RR$ for R as right, respectively left R-module and $\rho_a : x \mapsto xa$, $\lambda_a : x \mapsto ax$ for the right, respectively left multiplication by a, we have $\mathrm{End}_R({}_RR) \cong R$ via the map $a \mapsto \rho_a$ and $\mathrm{End}_R(R_R) \cong R^{\circ}$ via the map $a \mapsto \lambda_a$, where the opposite ring R° means that the R-endomorphisms of R_R form a ring anti-isomorphic to R (because $\lambda_{ab} = \lambda_b\lambda_a$). In this case equations (6) and (8) become

$$\mathrm{End}_R(R^n) \cong R_n, \quad \mathrm{End}_{R_n}(R^n) \cong R^{\circ}. \tag{9}$$

The row vectors $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots$ and the corresponding column vectors e_i^{T} form bases for R^n, ${}^nR$ respectively, as R-modules, called the *standard bases*.

Returning to the case of a general R-module M, we can summarize the relation between M and ${}^nM$ as follows.

Theorem 0.2.5. *Let R be a ring and M a left R-module with endomorphism ring E. Then ${}^nM$ may be regarded as an R_n-module in a natural way, with endomorphism ring E, and there is a lattice-isomorphism between the lattice of R-submodules of M and the lattice of R_n-submodules of ${}^nM$, in which (R, E)-bimodules correspond to (R_n, E)-bimodules.*

Proof. The first part is just a restatement of (8). To establish the isomorphism we recall that ${}^nM$ consists of columns of vectors over M; any submodule N of M corresponds to a submodule ${}^nN$ of ${}^nM$ and the correspondence

$$N \mapsto {}^nN \tag{10}$$

is order-preserving. Conversely, if P is an R_n-submodule of ${}^nM$, then the n projections $\pi_i : P \to M$ $(i = 1, \ldots, n)$ all have the same image and associate with P a submodule of M. The correspondence $P \mapsto P\pi_1$ easily seen to be an order-preserving map inverse to (10), hence (10) is an order-isomorphism between lattices, and so a lattice-isomorphism. The rest follows because the E-action on M and on ${}^nM$ is compatible with the R-action. ∎

The equivalence between R and R_n may be used to reduce any categorical question concerning a finitely generated module to a question for a cyclic module, over an appropriate ring. For, given M, generated as left R-module by $u_1, \ldots, u_n$, say, we apply the functor

$$M \mapsto {}^nM = \mathrm{Hom}_R({}^nR, M) = {}^nR \otimes_R M, \tag{11}$$

and pass to the left R_n-module ${}^nM$, which is generated by the single element $(u_1, \ldots, u_n)^{\mathrm{T}}$.

Thus we have proved

Theorem 0.2.6. *Any R-module M with an n-element generating set corresponds to a cyclic R_n-module under the equivalence (11).* ∎