Self-Organised Criticality

Theory, Models and Characterisation

Giving a detailed overview of the subject, this book takes in the results and methods that have arisen since the term 'self-organised criticality' was coined twenty years ago.

Providing an overview of numerical and analytical methods, from their theoretical foundation to actual application and implementation, the book is an easy access point to important results and sophisticated methods. Starting with the famous Bak–Tang–Wiesenfeld sandpile, ten key models are carefully defined, together with their results and applications. Comprehensive tables of numerical results are collected in one volume for the first time, making the information readily accessible to readers.

Written for graduate students and practising researchers in a range of disciplines, from physics and mathematics to biology, sociology, finance, medicine and engineering, the book gives a practical, hands-on approach throughout. Methods and results are applied in ways that will relate to the reader's own research.

Gunnar Pruessner is a Lecturer in Mathematical Physics in the Department of Mathematics at Imperial College London. His research ranges from complexity, through field theoretic methods and applications, to synchronisation and the application of statistical mechanics in the medical sciences.

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Theory, Models and Characterisation

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More Information

In memory of our friend Holger Bruhn.

To see the world in a grain of sand,
and to see heaven in a wild flower,
hold infinity in the palm of your hands,
and eternity in an hour.
William Blake, *Auguries of Innocence*, ca 1803

Contents

List of tables — *page* x
Foreword by Henrik J. Jensen — xi
Preface — xiii
List of symbols — xvii

Part I Introduction — 1

1 Introduction — 3
1.1 Reviews — 5
1.2 Basic ingredients — 7
1.3 Basic observables and observations — 11
1.4 Universality — 18

2 Scaling — 25
2.1 Distribution functions — 26
2.2 Moments — 37
2.3 Algebraic correlations — 46
2.4 Multiscaling — 50

3 Experiments and observations — 52
3.1 Granular media — 53
3.2 Superconductors — 58
3.3 Barkhausen effect — 62
3.4 Earthquakes — 66
3.5 Evolution — 68
3.6 Neural networks — 69
3.7 Other systems — 71
3.8 A very brief conclusion — 76

Part II Models and numerics — 79

4 Deterministic sandpiles — 85
4.1 The BAK–TANG–WIESENFELD Model — 85
4.2 Dhar's Abelian Sandpile Model — 91
4.3 The ZHANG Model — 104

5 Dissipative models 111
- 5.1 The Bak–Chen–Tang Forest Fire Model 112
- 5.2 The Drossel–Schwabl Forest Fire Model 115
- 5.3 The Olami–Feder–Christensen Model 125
- 5.4 The Bak–Sneppen Model 141

6 Stochastic sandpiles 162
- 6.1 The Manna Model 162
- 6.2 Dhar's Abelian Manna Model 166
- 6.3 The Oslo Model 183

7 Numerical methods and data analysis 210
- 7.1 Independence and correlations 211
- 7.2 Subsampling 215
- 7.3 Moment analysis 220
- 7.4 PDF estimation 229

Part III Theory 241

8 Analytical results 243
- 8.1 Random walkers and mean-field approaches 243
- 8.2 Renormalisation 265
- 8.3 Abelian symmetry and the operator approach 269
- 8.4 Directed models 285
- 8.5 Surfaces, interfaces, depinning and growth phenomena 298

9 Mechanisms 317
- 9.1 SOC and ordinary criticality 318
- 9.2 Dissipative transport in open systems 321
- 9.3 The AS mechanism 329
- 9.4 Other mechanisms 342

10 Summary and discussion 346
- 10.1 Convenient fallacies and inconvenient truths 348
- 10.2 Outlook 355

Appendix The Olami–Feder–Christensen Model in C 357
- A.1 Stacks 362
- A.2 Moments 363
- A.3 Boxes 365
- A.4 Main 369
- A.5 Output and termination 383
- A.6 Other techniques 384

Notes 391
References 398
Author index 459
Subject index 474

Tables

2.1 Scaling relations exemplified for the MANNA Model *page* 44
Overview of models – general characteristics 82
Overview of models – scaling 83
4.1 Exponents of the Abelian BTW Model 92
4.2 Exponents of the ZHANG Model 106
5.1 Exponents of the DS-FFM 120
5.2 Exponents of the non-conservative OFC Model, $\alpha < 1/q = 1/4$, open boundary conditions 128
5.3 Exponents of the conservative OFC Model, $\alpha = 1/q$, open boundary conditions 129
5.4 Exponents of the BS Model 143
6.1 Exponents of the AMM (and the oMM) 168
6.2 Exponents of the oOM (and the sOM where marked by $\diamond$) 189
8.1 Exponents of most directed sandpile models 289
8.2 Comparison of the exponents characterising the MANNA Model and the DP universality classes 294
8.3 Three different interface interpretations of SOC 300
8.4 Comparison of the exponents characterising the MANNA Model and the C-DP universality classes 306
8.5 Comparison of the exponents characterising the MANNA Model and the qEW equation 316
9.1 Comparison of $\beta^{(\mathrm{AS})}/\nu_\perp^{(\mathrm{AS})}$ in Lübeck (2004), based on Lübeck and Heger (2003b) and $d + \mathsf{z} - \mathsf{D}$ using the data in Table 6.1 340

Foreword by Henrik J. Jensen

When Self-Organised Criticality (SOC) was first introduced in 1987 by Bak, Tang, and Wiesenfeld, it was suggested to be *the* explanation of the fractal structures surrounding us everywhere in space and time. The very poetic intuitive appeal of the combination of terms self-organisation and criticality, meant that the field gained immediate attention. The excitement was not lowered much by the fact that the claimed $1/f$ and fractal behaviour were soon realised in reality not to be present in the sandpile model used by the authors to introduce their research agenda. Nor did the lack of power laws in experiments on real piles of sand deter investigators from interpreting pieces of power laws observed in various theoretical models and physical systems as evidence of SOC being essentially everywhere. This led rapidly to a strong polarisation between two camps. On the one side there was the group of researchers who did not worry about the lack of a reasonably precise exclusive definition of the SOC concept and therefore tended to use SOC as synonymous with snippets of power laws, rendering the term fairly meaningless. The other camp maintained that SOC was not to be taken seriously. They arrived at this conclusion through a mixture of factors including the observation that SOC was ill defined, not demonstrated convincingly in models, and absent from experiments on sandpiles. The debate sometimes reflected a reaction in response to bruises received during fierce exchanges at meetings as much as a reaction to scientific evidence.

All in all this was a difficult, somewhat overwrought and unfortunate situation. Science proceeds through gradual uncovering of hierarchical insights and it is of course most unlikely that one simple mechanism is able to explain *how nature works*. Nevertheless, we are certainly surrounded by abundant power laws and fractals of more or less pure forms. They are typical features of non-equilibrium systems. But we obviously need to go beyond the classification of the world into only two categories: equilibrium and non-equlibirum. There is structure among the non-equilibrium systems – exactly as the classification of zoology into elephants and non-elephants does not represent the rich substructures within the class of non-elephants, such as tigers, parrots, monkeys and tortoises, to mention a few. When SOC is scrutinised at a seriously scientific level it will most definitely help us in our quest to understand the substructures within the non-equilibrium class.

The present book does exactly this. It seeks to lay out carefully what is certain and the uncertain aspects about SOC. Gunnar Pruessner has produced a comprehensive, detailed and authoritative overview of a quarter of a century's intensive experimental, observational and theoretical investigations. He carefully discusses the main achievements, the still unsolved problems and some of the compelling reasons why SOC remains an important concept helping us to place some demarcation lines within the vast field of non-equilibrium systems. The book is a masterpiece in clarity surveying a huge literature and helping to extract the essential durable lessons uncovered by the many scientists working in the field. The book will play an essential rôle in turning SOC into a serious and specific research field. It is worth mentioning how the book compares with Bak's (1996) *How Nature Works* and with my own (1998) book, *Self-Organized Criticality*. Bak's book is not really scientific. It is an entertaining and readable, enthusiastic and very optimistic document about how SOC came into being, what Bak hoped it would achieve and how Bak experienced fellow researchers in the field. My own book was a very brief and simple attempt to make a status report of what SOC might mean scientifically after the first 10 years of research. Not only does Pruessner's book present 15 more years of research, it also represents a significantly more detailed and mathematically careful discussion.

Like many other fields of statistical mechanics SOC now has its own reference book that helps us as we continue to develop our understanding of what SOC really is – and is not. By the clear exposition of the established understanding of models and the relationships between them, the book will help research aimed at developing further our theoretical comprehension. Furthermore, by the circumspect discussion of the relation between theoretical insights and expectations on the one side and experiments and observations on the other, the book will act as a guide when we continue our attempts to place experimental and observational findings from physics, biology, neuroscience, geophysics, astrophysics, sociology etc. in relation to SOC.

Henrik Jeldtoft Jensen

Preface

Self-organised criticality (SOC) is a very lively field that in recent years has branched out into many different areas and contributed immensely to the understanding of critical phenomena in nature. Since its discovery in 1987, it has been one of the most active and influential fields in statistical mechanics. It has found innumerable applications in a large variety of fields, such as physics, chemistry, medicine, sociology, linguistics, to name but a few. A lot of progress has been made over the last 20 years in understanding the phenomenology of SOC and its causes. During this time, many of the original concepts have been revised a number of times, and some, such as complexity and emergence, are still very actively discussed. Nevertheless, some if not most of the original questions remain unanswered. Is SOC ubiquitous? How does it work?

As the field matured and reached a widening audience, the demand for a summary or a commented review grew. When Professor Henrik J. Jensen asked me to write an updated version of his book on self-organised criticality six years ago, it struck me as a great honour, but an equally great challenge. His book is widely regarded as a wonderfully concise, well-written introduction to the field. After more than 24 years since its conception, self-organised criticality is in a process of consolidation, which an up-to-date review has to appreciate just as much as the many new results discovered and the new directions explored. Very soon the idea was born to deviate from Henrik's plan and try to be more comprehensive. It proved a formidable task to include all the material I thought deserved mentioning, and a daunting and precarious one to decide what not to include. There is a degree of uncertainty that I did not expect and for which I sincerely apologise.

Leaving all difficulties aside, SOC is regarded in the following as *scale invariance without external tuning of a control parameter, but with all the features of the critical point of an ordinary phase transition, in particular long range (algebraic) spatiotemporal correlations*. In contrast to (other) instances of generic scale invariance it displays a separation of time scales, avalanches, is interaction dominated and contains non-linearities. The ideal SOC model differs from a corresponding model displaying an ordinary phase transition solely by a self-tuning mechanism, which is absent in the latter. Defining SOC primarily as a phenomenon rather than a class of systems allows the question which natural phenomena, experiments, computer models and theories display the desired behaviour. This is the central theme of the present book. It was written with the aim of giving an overview of the vast literature, the many theories and disparate methods used to analyse SOC phenomena. It is my hope that it will be of as much use for the theoretician working in statistical mechanics, as for the practitioner trying to apply ideas of SOC to his or her particular field.

Style

With this broad scope, I have tried to make the material as accessible as possible, by placing it in a wider context or a narrative, whenever suitable, by retracing the historic development of the field. I have aimed for a balanced view on contentious questions and have tried to flag up where my personal opinion and convictions have taken over. I hope the occasional casual remark is never mistaken as an offence. I have tried to use notation, symbols and names consistently throughout, but only as long as this did not clash with established usage. Where I could not establish a particular custom, I used the notation of the original works. A table of commonly used symbols can be found on pp. xvii–xxii.

Structure

The book is divided in three parts. The first part gives an introductory overview of the subject: the basic concepts and terminology, the principles of scaling and the huge range of experimental evidence.

In the second part, the key SOC models [1] are discussed, divided in three categories: sandpile-like, deterministic models, dissipative models and stochastic sandpiles. I have tried to pick models that have received a large amount of attention and had a great impact on the field as a whole. These difficult choices are, admittedly, arbitrary and thus bound to be imperfect. Each model is discussed in some detail, starting with a short history followed by

> its definition, as well as its key features and observables set apart in a Box, which provides the reader with an overview of the dynamics of the model.

Because every such box follows the same structure, they can serve as a reference and are easily turned into a concrete implementation.

After a few introductory remarks, the remainder is dedicated to the particular characteristics of the model, focusing on those that have evoked the most wide-ranging discourse in the literature. The material has been brought together with the intention of covering as much of the relevant findings as possible, even when they are sometimes contradictory. The last chapter of the second part is dedicated to various numerical techniques used for the characterisation and analysis of SOC models. I mention implementation details for particular models whenever I see fit, unfortunately I was not able to present a complete computer program for each model. In the appendix, an actual C implementation of an SOC model is discussed in great detail using the example of the OFC Model.

[1] To call these systems 'models', as is traditionally done, is misleading if one accepts that they are not intended to model much more than themselves. A similar concern applies to the term 'simulation', see footnote 1 on p. 210.

The final, third part reviews many theoretical concepts, such as mean-field theories and random walker approaches, which have proved invaluable for the analysis of SOC models. Such theories have led to various proposed mechanisms, which are presented in Ch. 9. Again, I had to make an arbitrary choice and decided to focus on two theories of SOC, summarising others only very briefly. The very last chapter might be the most interesting for many readers and might even prove a bit contentious. Here I have collected facts and fallacies, as ignored and embraced in the literature (although I have refrained from stating bibliographical references). From my point of view, such erroneous beliefs are remarkably instructive and do more good than harm. As the chapter draws much of its arguments from the preceding ones, one might regard it as a form of summary – the book ends with a very brief outlook.

Inevitably, I myself suffer from such fallacies, ignore the truth, misinterpret findings, overrate my own work and overlook important contributions by others. I have tried very hard to avoid such mistakes and I sincerely apologise for the failure to do so. I would like to invite readers to share their insights with me and contact me whenever corrections are due.

Keywords, notes and indices

Throughout the text, important **concepts and keywords** are shown in bold, as are their page numbers in the index. *Key statements* are printed in italics, and slanted text in a figure caption refers to a particular feature in the figure. Footnotes are indicated by arabic superscript numbers.[2] Where more extensive notes could not be presented in a footnote, endnotes are used, marked in the text by a number in square brackets.[0.0] Endnotes can be found on pp. 391–398, just before the references. Each is preceded by the page number referring to it.

The accessibility of the material is facilitated by the indices in the back of the book. Each bibliographical entry contains, after an arrow (→), the page numbers where the work is cited, with italics signifying *quotations*. Publications that deserve special attention are marked as follows: ● marks selected reviews and overview articles, ▶ designates the definition of some characteristic or important models and ⇒indicates a proposed mechanism of SOC. To simplify sequencing, in all indexes name prefixes, such as 'van' and 'de', are regarded part of the name, for example 'van Kampen' can be found under 'V'. The references are followed by an author index and a subject index. In the former, bold page numbers refer to **bibliographical entries** by that author and italicised ones, again, refer to *quotes*. The subject index is rather detailed and has a large number of entries; where <u>entire sections</u> are dedicated to a subject, the page numbers are underlined. Page numbers of **keywords** are highlighted in bold. **Key SOC models**, discussed in Part II of the book, are printed in bold. EPONYMS and NAMESAKES of SOC models are highlighted in small capitals throughout. I have used established abbreviations and names as much as possible, but inevitably some authors use different ones.

[2] See footnote 1 on p. xiv for an example.

Numerics

A pitfall Henrik Jensen wisely avoided in his introduction to SOC is the inclusion of numerical results. What should be included? The most up-to-date results, which soon will be out-dated? The historic values, which have been built upon? I decided to include as much as possible and necessary to allow the reader to judge which results can be relied upon. Unfortunately, it is not always immediately clear which numbers have been derived from numerical data under which assumptions. I have tried to avoid making additional ones. Where it occurred to me as being particularly important, I point out that exponents have been derived from each other, not least because that puts an apparent consistency in perspective.

Acknowledgments

I would like to thank Henrik J. Jensen for his kindness and sincerity and the opportunity he has given me by suggesting that I write this book. I am deeply grateful for all he taught me about science and SOC in particular and look forward to the many projects we enjoy together. This is what Science should be like. I am indebted to Kim Christensen, Nicholas Moloney, Andy Parry and Ole Peters, who allowed me to learn so much in the many discussions and collaborations we have had. I would like to extend my gratitude to everyone who helped me to write this book and contributed to it. Attempting to name everybody is a task bound to fail and I apologise for that. People I would like to mention, because they so kindly shared their insights with me include: Alvin Chua, Deepak Dhar, Ronald Dickman, Vidar Frette, Peter Grassberger, Christopher Henley, Satya Majumdar, Robert McKay, Zoltan Racz, Richard Rittel, Beate Schmittmann, Paolo Sibani, Alessandro Vespignani, Alistair Windus and Royce Zia. I would also like to thank my friends at Imperial College London, in particular Seng Cheang, Nguyen Huynh and Ed Sherman, who make working here such a wonderful experience. We have the good fortune of a great library staff, who tirelessly tracked down and provided me with the most elusive material. I am truly grateful for their efforts. The same goes for our fantastic computing support, in particular Dan Moore and Andy Thomas, who provided me with a seemingly endless amount of CPU time, even when the demands of modern university life make that all the more difficult.

The cover of this book shows ‘Work No. 370: Balls’ by M. Creed (2004), which is a stunning example for the need of a reference scale to judge correctly the size of the objects shown – without the parquet in the background it would be difficult to tell whether the baseball or football is out of proportion. I am very grateful to the artist and the gallery Hauser and Wirth for giving me access to this wonderful work.

Finally, I would like to thank Anton, Ida and Sonja without whom nothing would be much fun. It is impossible to make up for all the support and patience I received from them, as I did from our friends and family, in particular from both our parents. One day I wish to have the opportunity to pass on to my children all this kind, patient help to build a home, a family and a life. This book is dedicated to our friend Holger Bruhn, who is so greatly missed.

Symbols

An attempt was made to use symbols consistently throughout the book, at the same time keeping them in line with common usage and tradition as much as possible. Most of them are carefully introduced when they are used for the first time in a chapter. A few have more than one or a more specific meaning, depending on the particular context, or are used briefly for a very different purpose than the one listed below. Many have derived forms with a more specialised meaning not listed below. The page numbers refer to the earliest occurrence of the variable or its most detailed discussion. Further references can be found in the index.

Symbol	Description	Page
α	Avalanche duration exponent in SOC models	14
α	Scaling exponent of the power spectrum ($1/f^{\alpha}$ noise)	15
α	Level of conservation in the OFC Model	126
α_E	Scaling exponent of the scaling function of the energy dissipated in the ricepile experiment	191
α_R	Asymptotic power law of the residence time distribution in the OSLO Model	195
β	Order parameter exponent in ordinary critical phenomena	337
β_E	Finite size scaling exponent of the energy dissipated in the ricepile experiment	191
β_R	Finite size scaling exponent of the residence time distribution in the OSLO Model	194
$\beta^{(\mathrm{int})}$	Growth (roughening) exponent of an interface *Superscripts in brackets generally denote the observable the variable characterises.*	310
$\beta^{(\mathrm{AS})}$	Order parameter exponent in an absorbing state phase transition	337
Γ	Noise amplitude (of various types of noise)	205
$\Gamma(\cdot)$	Gamma function	57
γ	Critical exponent of the average avalanche size in the BS Model	148
γ	Susceptibility exponent in ordinary critical phenomena	337
γ_{xy}	Exponent characterising scaling of observable x conditional to y	42
$\gamma^{(\mathrm{DP})}$	Susceptibility exponent in directed percolation	297
Δp	Distance from the critical point in percolation (and similar for other variables)	30

Symbol	Description	Page
$\Delta_{\mathbf{nm}}$	Toppling matrix	91
$\Delta(g)$	Correlator of the noise in the quenched Edwards–Wilkinson equation	205
δ_{ij}, $\delta(\cdot)$	Kronecker and Dirac δ function	
$\delta^{(\mathrm{AS})}$	Survival exponent in absorbing state phase transitions	305
ϵ	Small quantity, correction etc.	
ϵ	Dissipation rate (often in the bulk)	169
ϵ	Mass term	323
ζ	Particle density	330
η	Correlation exponent	47
θ	Tuning variable in the DS-FFM	116
θ_m, θ_r	Angle at which avalanching must occur, angle of repose	54
$\theta(\cdot)$	Heaviside step function	32
Λ	Total rate of multiple Poisson processes	389
λ	Scaling factor	32
λ	Scaling exponent of the characteristic burnt cluster size in the DS-FFM	119
λ	Coupling of the non-linearity	324
λ	Control parameter in the contact process	341
$\lambda, \lambda_i, \ldots$	Rate of a Poisson process	388
λ_i	ith eigenvalue of eigenvector $\|e_i\rangle$	201
μ	Generic scaling exponent	28
ν	Surface tension	47
ν	Correlation length exponent in the FFM	112
ν	Correlation length exponent in ordinary critical phenomena	337
ν'	Scaling of the characteristic burning duration in the DS-FFM	118
ν_E	Finite size scaling exponent of the energy dissipated in the ricepile experiment	191
ν_R	Finite size scaling exponent of the residence time distribution in the Oslo Model	194
$\nu_\perp^{(\mathrm{AS})}$	Correlation length exponent in absorbing state phase transitions	305
ξ	Correlation length	112
$\xi, \tilde{\xi}$	Noise (usually Gaussian, white, vanishing mean)	204
ρ	Tree density in the DS-FFM	118
ρ_n	Moment ratio	40
ρ_a	Activity	299
Σ	Finite size scaling exponent	43
σ	Branching ratio (effective)	130
σ	Critical exponent of the characteristic avalanche size in the BS Model	160
$\sigma^2(\cdot)$	Variance	

Symbol	Description	Page
$\overline{\sigma}^2\,(\cdot)$	Numerical estimate of the variance *Overbars generally denote numerical estimates.*	
$\sigma_n^{(s)}, \sigma_b^{(T)}$	Finite size scaling exponent of the nth moment of the avalanche size (or duration) distribution	38
τ	Avalanche size exponent in SOC models	13
τ	Correlation time on the microscopic time scale	212
$\tilde{\tau}$	Apparent avalanche size exponent in SOC models	31
τ_r	Avalanche radius exponent	43
Φ	Number of potential topplings in the OSLO Model	202
$\chi^{(\mathrm{pile})}$	Roughness exponent of the surface of a sandpile	197
$\chi^{(\mathrm{int})}$	Roughness exponent of an interface	308
$\Omega(f(x))$	Asymptotically bounded from below by $f(x)$	292
ω	Frequency (angular velocity)	15
$\omega, \omega_1, \ldots$	Correction to scaling exponent	30
A	Avalanche area	12
$A, A_n, B, \ldots$	Generic amplitudes	31
A_h, A_t	Areas two distributions differ by	34
$a, a(t)$	Number of active sites, position of a random walker	252
$a^{(s)}, b^{(s)}, \ldots$	Metric factors	12
$a(t)$	Time series	15
a_0	Initial distance of a random walker from an absorbing wall	36
$a_{\mathbf{n}}$	Evolution operator in deterministic sandpiles	95
$\hat{a}_{\mathbf{n}}$	Markov matrix	272
b	Exponent in the Gutenberg–Richter law	66
b_j	Bin boundary	231
$\mathcal{C}$	Stable configuration (state) on the lattice	95
$C, C^\dagger, C_{\mathbf{n}} \ldots$	Creation and annihilation operators in the MANNA Model (matrix representation of its dynamics), acting on site $\mathbf{n}$	173
$\mathcal{C}(\cdot)$	Cumulant generating function	46
C_t	Catalan number	253
$c(t)$	Time correlation function	15
$c(\mathbf{x}, \mathbf{x}'\, t, t')$	Connected correlation function	47
$c_{\mathbf{n}}$	Probability for site $\mathbf{n}$ to be charged	97
c_0	Density of empty sites in the sticky sandpile	295
D	Avalanche dimension (finite size scaling exponent) in SOC models	12
D	Diffusion constant	36
$D_t(t)$	Probability for the activity to cease at time t	253
d	Spatial dimension	27
E	Energy release in an earthquake	66
E	Energy release in the ricepile experiment	191
$E_{\mathbf{n}}$	Energy at site $\mathbf{n}$ in the ZHANG Model	105

Symbol	Description	Page
$E(t), E(\mathbf{n}, t)$	External drive (boundary condition or source term) in the OSLO Model	205
$\langle e_i\| , \|e_i\rangle , \ldots$	Left and right eigenvectors, respectively, with eigenvalue λ_i	201
$\mathcal{F}, \mathcal{G}, \tilde{\mathcal{G}}, \mathcal{K}, \ldots$	Scaling and cutoff functions	12
$\mathsf{F}(\cdot)$	Cumulative distribution function	26
$F_\mathbf{n}, F_\mathbf{th}$	Force at **n** and threshold in the OFC Model	127
f	Frequency	15
f	Lightning probability in the DS-FFM	116
$f(x), g(x)$	Generic function or functional	
$f(s; s_\mathbf{c})$	Non-universal part of the probability density function of s	28
$f_n^{(s)}$	Moment of the non-universal part of the probability density function	29
$f_\mathbf{n}$	Fitness at site **n** in the BS Model	142
f_0	Fitness threshold to define avalanches in the BS Model	142
$G(t)$	Gap function in the BS Model	144
$G(\mathbf{n}, t)$	Number of topplings performed by site **n** up to and including time t	176
$g_n^{(s)}$	Moment of the scaling function	29
$g(\mathbf{n}, t)$	Number of charges received by site **n** up to and including time t	176
$g, g_\mathbf{n}$	Generational time scale, local generational time scale	257
$H(\mathbf{x}; l)$	Coarse grained local dynamical variable	49
h	External driving field in the AS mechanism	331
$h(\mathbf{x}, t)$	Height field in continuum, generally local dynamical variable	47
$h_\mathbf{n}$	Height of particle pile or particle number at lattice site **n** in BTW and MANNA Models	86
h^c	Critical height or particle number in the MANNA Model	164
h_i	Normalised histogram count for slot i	230
$I_n(\mathbf{n})$	Indicator function of site **n** being in state n	99
$i, j, \ldots$	Generic indices and counters	
$\imath$	Imaginary unit, $\sqrt{-1}$	
$\mathbf{j}$	Current	324
K	Number of new fitnesses drawn per site updated in the BS Model	392
$\mathcal{K}(\cdot)$	Scaling function of the two point correlation function	47
K_L	Spring constant in the OFC Model	125
$\mathbf{k}$	Generic vector in Fourier space	
L	Linear system size	86
l	Block size	49
$M, N, \ldots$	Generic number of items (N is often the number of sites)	
$\mathcal{M}(\cdot)$	Moment generating function	45
$\mathbf{m}, \mathbf{n}, \ldots$	Generic position on the lattice	
m_i	Arithmetic mean of a subsample	216
$\langle m\rangle (\mathbf{n})$	Average number of moves (escape time) from **n**	245
$m(g)$	Size of the gth generation in a branching process	258

Symbol	Description	Page
$N_i, \mathcal{N}$	Size of a subsample, total sample size	216
$n(s)$	Site normalised cluster size distribution	103
n_{cov}	Number of distinct sites updated in the BS Model (avalanche area)	392
$\mathcal{O}(f(x))$	Asymptotically bounded from above by $f(x)$	
$P(\cdot)$	Generic probability of an event to occur, a quantity (usually discrete) to be observed etc.	
$\mathcal{P}^{(s)}(s')$	Probability density function of observable s evaluated at s'	12
p	Number of boundary sites toppling in an avalanche	42
p	Period of an Abelian operator	95
p	Tree growth probability in the FFM	113
p	Parameter in the ricepile model	185
p	Toppling probability in the sticky sandpile	295
p_r, p_l, p_0	Toppling probabilities in the generalised AMM	174
$p_c^{(\mathrm{DP})}$	Directed percolation critical point	295
Q	Linear observable	275
q	Coordination number on a lattice	
$\tilde{q}$	Removals per toppling	245
$q'(\mathbf{x})$	Local ratio of number of charges and topplings	307
R	Residence time in the Oslo Model	186
R	Binning range when coarse graining	233
$R_{1/2}$	Characteristic length for the decay of correlations	48
r	Radius of gyration	12
r, r_0	Distance between two points in space	48
r_i	Binning range	230
S	Total area encapsulated in two interface representations of an Oslo Model configuration	395
$S(\omega)$	Power spectrum	15
S_1, S_2	Parameter in the ricepile model	185
$S, S_{\mathbf{n}}, T, \dots$	Evolution matrices acting on site $\mathbf{n}$ in the matrix representation of the Abelian BTW Model and the (totally asymmetric) Oslo Model	200
S_N	Sum of the sizes of N consecutive avalanches	212
s	Avalanche size	11
s_0	Lower cutoff of the avalanche size distribution	12
s_{c}	Upper cutoff of the avalanche size distribution (characteristic size)	12
T	Avalanche duration	14
T	Number of subsamples	216
T	Longitudinal direction in the DR Model	286
T_0	Lower cutoff of the avalanche duration distribution	14
T_{c}	Upper cutoff of the avalanche duration distribution (characteristic time scale)	14
T_t, T_g	Termination probability (random walk and branching process)	253
$\mathcal{T}(\mathcal{C} \to \mathcal{C}')$	Probability for a transition from $\mathcal{C}$ to $\mathcal{C}'$	273

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Gunnar Pruessner
Frontmatter
More Information

Symbol	Description	Page
t	Generic time	
$\mathcal{U}(\cdot)$	Diagonal matrix generating the potential number of topplings in the Oslo Model	202
$u(s_{\mathbf{0}}; s_{\mathbf{c}})$	Fraction of events governed by the universal part of the probability density function	29
V	Volume of the system	118
v	Drift velocity	206
W	Double charging Markov matrix of the AMM	174
w, w^2	Width and roughness of an interface, surface etc.	313
x, y	Generic real variable, components of $\mathbf{x}$	
$\mathbf{x}$	Generic position in continuum space	
z	Dynamical exponent in SOC models	14
$z_{\mathbf{n}}$	Slope at $\mathbf{n}$	86
z^c, $z^c_{\mathbf{n}}$	Critical slope, at site $\mathbf{n}$	87
$z^{(\mathrm{AS})}$	Dynamical exponent in absorbing state phase transitions	305
$z^{(\mathrm{int})}$	Dynamical exponent in interface growth phenomena	309