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1 Introduction

When Bak, Tang, and Wiesenfeld (1987) coined the term **Self-Organised Criticality** (SOC), it was an explanation for an unexpected observation of scale invariance and, at the same time, a programme of further research. Over the years it developed into a subject area which is concerned mostly with the analysis of computer models that display a form of **generic scale invariance**. The primacy of the **computer model** is manifest in the first publication and throughout the history of SOC, which evolved with and revolved around such computer models. That has led to a plethora of computer 'models', many of which are not intended to model much except themselves (also Gisiger, 2001), in the hope that they display a certain aspect of SOC in a particularly clear way.

The question whether SOC exists is empty if SOC is merely the title for a certain class of computer models. In the following, the term SOC will therefore be used in its original meaning (Bak *et al.*, 1987), to be assigned to systems

> with spatial degrees of freedom [which] naturally evolve into a self-organized critical point.

Such behaviour is to be juxtaposed to the traditional notion of a **phase transition**, which is the singular, **critical point** in a phase diagram, where a system experiences a breakdown of symmetry and **long-range spatial** and, in non-equilibrium, also **temporal correlations**, generally summarised as (power law) **scaling** (Widom, 1965a,b; Stanley, 1971). The paradigmatic example for such a **critical phenomenon** is the Ising Model, which displays scaling only at a specific critical temperature, the value of which depends on the type and dimension of the lattice considered, as well as the details of the interaction, and is generally not known analytically.

Bak *et al.* (1987) found a simple computer model which seemed to develop into a non-trivial scale invariant state, displaying long-range spatiotemporal correlations and self-similarity, without the need of any tuning of a temperature-like control parameter to a critical value. **Universality** of the asymptote of the correlation function is immediately suspected by analogy with ordinary scale invariance, i.e. critical phenomena. More than twenty years later, one can review the situation. Does SOC exist? A host of models has been studied in great detail with mixed results. While some eventually turned out not to display scaling, a range of models displays the expected behaviour. The situation is less encouraging for experimental evidence, which suggests that there are very few systems with solid scaling behaviour. The most pessimistic perspective on the numerical and experimental evidence is that none of the systems displays asymptotic scaling behaviour and SOC does not exist. With the theoretical, numerical and experimental evidence presented in the following, this point of view is difficult to maintain. SOC exists, very convincingly, in flux avalanches in superconductors and certainly in the MANNA Model and the OSLO Model. Yet, it is probably not as common as originally envisaged so that the enormous body of work reporting long-range behaviour, which is close to but not exactly a power law, still awaits an explanation beyond SOC.

There is a number of reasons why SOC is important independent of how broadly it applies. Initially, it was a solution of the riddle of $1/f$ noise (Sec. 1.3.2), the frequently found scaling of the **power spectrum** of a time series. Press (1978) popularised the notion of $1/f$ noise and its ubiquity (also Dutta and Horn, 1981; Hooge, Kleinpenning, and Vandamme, 1981; Weissman, 1988). SOC provided a promising explanation, which was long overdue. Perhaps more importantly, it answered Kadanoff's (1986) call for a 'physics of fractals' with a theory that epitomises Anderson's (1972) credo 'more is different': 'The aim of the science of self-organized criticality is to yield insight into the fundamental question of why nature is complex, not simple, as the laws of physics imply' (Bak, 1996, p. xi). As Gisiger (2001) summarised: with the advent of SOC '[. . .] the important question which arose from the work of Mandelbrot [. . .] has shifted from "Why is there scale invariance in nature?" to "Is nature critical?" [. . .]'. Thirdly and more concretely, if SOC were generally to be found across a large class of systems, then **universality** would unify these systems, and their internal interactions could be identified and studied. They dominate the long time and large scale behaviour, just like in ordinary critical phenomena with a broken symmetry and could be studied in a simplified experiment on the laboratory scale or a simple computer model, rather than 'attempt[ing] to model the detailed, and perhaps insuperably complex[,] microphysics' (Dendy, Helander, and Tagger, 1999). Even if universality does not apply to SOC, scaling still means that some large scale phenomena can be studied on the laboratory scale, provided only that both are governed by the asymptotic behaviour of the system. Finally, even if SOC is less common than initially expected, it might still help to elucidate the nature of the critical state in traditional critical phenomena. It might even serve as a recipe to make these systems self-tune to a transition.

To appreciate its impact, it is interesting to retrace the historical context of SOC. In the 1960s statistical mechanics gained the ability to incorporate large fluctuations in a meaningful way and moved from the physics of gases and liquids with small, local perturbations, to the physics of long-range correlations as observed at phase transitions. Kadanoff (1966) explained the scaling ideas brought forward by Ben Widom using the concept of renormalisation. In the 1970s Wilson's renormalisation group led to a deep understanding of phase transitions and symmetry breaking. The 1980s made fractals popular; their physical manifestations are collected in Mandelbrot's (1983) famous 'manifesto and casebook'. By the end of that decade, SOC was born and helped to promote the notion of complexity. Since then, the focus has shifted to the more general idea of **emergence**, which summarises in a single word the phenomenon that many interacting degrees of freedom can bring about **cooperative phenomena**, i.e. the whole is more than the sum of its parts and effective long-range interaction looks fundamentally different from the microscopic interaction it is caused by.

Like the 1980s theme of fractals, SOC reaches far beyond its original realm. Other subject areas that tap into the results in SOC often regard it as an *explanation* for the observed emergent phenomenon, as originally envisaged by Bak *et al.* (1987). Given, however, that SOC itself is not fully understood, its explanatory power stretches only so far as to assume that a given *scale-free phenomenon is caused by the system* ***self-organising*** *to an underlying critical point.* Whether that leads to any further insight depends on the nature of the phenomenon in question. For example, if evolution is self-organised critical (Sec. 3.5), the hunt for external causes of mass extinction is put into question. Claiming that the Barkhausen effect (Sec. 3.3) is self-organised critical adds much less to the understanding of the phenomenon and might ultimately be seen merely as a change of perspective.

Apart from identifying systems that indisputably display SOC, a lot of research is dedicated to the question how SOC works, i.e. its necessary and sufficient conditions. Among the many proposed mechanisms, the Absorbing State Mechanism (AS mechanism, Sec. 9.3) has gained so much popularity that SOC is considered by some as explained. The fact that many of its key ideas were laid out already by Bak *et al.* (1987, also Tang and Bak, 1988b) is a striking testimony of their genius. According to the AS mechanism, SOC is due to a very simple feedback loop, leading to the **self-tuning** of a parameter that controls a non-equilibrium phase transition to its critical point. As discussed in detail elsewhere (Sec. 9.3.4), some important ingredients of the AS mechanism might still be missing.

The big challenges in SOC on a more technical level thus have remained the same for almost twenty years: on the most basic level, the identification of universality classes containing models that display solid scaling behaviour. This is mostly numerical work and significant progress has been made in particular with respect to the Manna universality class (Sec. 6.2.1.2). Other universality classes displaying robust scaling are very scarce, virtually non-existent. A much bigger challenge is to develop a full understanding of the underlying mechanism including the link to ordinary critical phenomena, which includes absorbing state phase transitions. Such a mechanism might be applicable to ordinary critical phenomena. Finally, the big question: why are so many natural phenomena so broadly distributed, (almost) resembling a power law? The answer to this question might, of course, lie far outside the realm of SOC.

The questions where SOC can be found, which systems are governed by it, what that implies, what the general features of SOC are, its necessary and sufficient conditions, all of this is still a very active research field. After a short overview of other reviews on SOC, in the remainder of this chapter the key concepts of SOC are introduced as well as its basic ingredients and observables. Two more chapters in Part I introduce the concepts and technicalities of scaling and a review of the many SOC experiments and observations. Part II is dedicated to a number of widely studied (computer) models displaying SOC, while Part III discusses various analytical approaches to the understanding of SOC.

1.1 Reviews

Over the years, a large number of reviews of SOC has been published. Of those with very broad scope, Bak's (1996) famous book with the equally ambitious as teasing title 'how nature works', popularised the subject, but also antagonised a few. The very readable, succinct review by Jensen (1998) soon became the most cited academic reference, providing a comprehensive and comprehensible overview of the main results and central themes in SOC. Sornette's (2nd edition 2006) book places SOC in a broader context with a wider background in probability theory and physics. Christensen and Moloney's (2005) textbook takes a similar approach and is aimed mainly at the audience of researchers in other subjects as well as undergraduates. A light and much shorter introduction can be found in the beautifully illustrated article by Bak and Chen (1991) and an even shorter one in Bak and Tang (1989b), which is probably relevant only for historic reasons. A broad but more technical overview is given by Bak and Paczuski (1993) and Bak and Paczuski (1995), which place SOC in the context of complexity and vice versa (Vicsek, 2002). As for the relation between complexity and statistical mechanics, Kadanoff (2000) is an invaluable collection of original articles and commentary.

Shorter, often more specialised and much more technical, but also more up-to-date reviews have frequently been published over the last fifteen years or so. Dhar (1999a,c, 2006) concentrated mostly on exact results, whereas Alava (2004) discussed the relation of SOC to (ordinary) non-equilibrium phase transitions, growth phenomena and interfaces in random media. This approach can be traced back to the highly influential review by Dickman, Muñoz, Vespignani, and Zapperi (2000), which presented some of the key models and experiments on SOC in the light of absorbing state phase transitions. Volume 340, issue 4 of *Physica A* (Alstrøm, Bohr, Christensen, *et al.*, 2004) contains short and very diverse reviews from the participants of a symposium in memory of Per Bak, a valuable resource to retrace the history of SOC and its impact on current research. At least two other proceedings volumes are similarly useful. The first half of the volume edited by Riste and Sherrington (1991) gives an overview of SOC and its context only a few years after its conception. The volume edited by McKane, Droz, Vannimenus, and Wolf (1995), which contains Grinstein's (1995) highly influential review, shows a more mature field, with links to growth, generic scale invariance and cellular automata. An early, detailed and quite technical review of SOC with special attention to extremal dynamics and interface depinning was published by Paczuski, Maslov, and Bak (1996). The recent development regarding the relation between SOC and absorbing states is discussed in Muñoz, Dickman, Pastor-Satorras *et al.* (2001). On the more applied side, Hergarten (2002) published a monograph on SOC in earth systems.

A highly critical assessment of SOC on a philosophical level can be found in Frigg (2003), who also presented an introduction to some of the frequently used SOC models. The epistomology of emergence and thus complexity is discussed in Batterman (2002). Earlier, Horgan (1995) wrote a more entertaining piece on complexity, including SOC, questioning to what extent it is delivering on the promises it apparently made (but Vicsek, 2002). Complexity might have replaced the oversimplified interpretation of reductionism as a way of reconstructing the world, as criticised by Anderson (1972), by an equally naïve notion of universality (Batterman, 2002), which takes away too many details of the phenomena science and technology are interested in. As much as psychology is not applied biology and chemistry is not applied particle physics, the universal phenomena shared by them might be pretty meaningless to them all.

1.2 Basic ingredients

A precise definition of the term 'self-organised criticality' is hampered by its indiscriminant use, which has blurred its meaning. Sometimes it is used to label models and natural systems with particular design features, sometimes for models with a particular (statistical) phenomenology, sometimes for the phenomenon itself. The literal meaning of the term refers to its supposed origin: a critical, i.e. scale invariant, phenomenon in a system with many degrees of freedom reminiscent of a phase transition, yet not triggered by gentle tuning of a **temperature-like control parameter** [1] to the critical point, as for example in the Ising Model, but by **self-organisation** to that critical point. This self-organisation might involve a control parameter which is subject to an equation of motion, or it might affect directly the statistical ensemble sampled by the system.

As SOC became more popular, the meaning of SOC switched from labelling the cause to labelling the phenomenon, i.e. scale invariance, characterised by power laws in spatiotemporal observables and their distributions. [2] As SOC was intended to explain $1/f$ noise, the latter became indicative or even synonymous with SOC. This is the beginning of a time when the question 'Is it SOC?' became meaningless (Sornette, 1994). It broke down completely when it became common to label anything SOC that bore some resemblance to its typical features, such as avalanches, thresholds or simply a broad distribution of some observable, regardless of whether or not there was any signature of scale invariant behaviour – some disputes in the literature are down to this simple confusion, as not all SOC models would display SOC.

It can be difficult to keep these different perspectives on SOC apart. Ideally, SOC should be reserved for the first meaning, a supposed underlying mechanism which keeps a system at or drives it to a (more or less) ordinary critical point. There is no need to apply the term SOC to the statistical phenomena, since plenty of terminology from the theory of phase transitions is available. Characterising certain models as 'SOC models', however, is so undeniably widespread that it would be unrealistic to propose to bar the usage of the term here, on the understanding that some 'SOC models' might ultimately not be governed by SOC.

There are a few basic ingredients that can be found in every SOC model, summarised by Jensen (1998, p. 126) as 'slowly driven, interaction-dominated threshold [(SDIDT)] systems'. First of all, there are **many (discrete) interacting degrees of freedom**, usually structured in space by nearest neighbour or at least local interaction, for example as **sites** on a lattice. Apart from some exotic exceptions, all models have a **finite number of degrees of freedom**, most suitably captured in the **finite size** of the lattice. Secondly, the interaction involves a **threshold**, which represents a very strong non-linearity. In many models the degrees of freedom can be thought of as a **local (dynamical) variable** indicating the amount of local **energy, force or particles**, which is redistributed among neighbours once it exceeds a threshold. In other models, sites interact in a certain way only if they are in

[1] Sometimes called the 'tuning parameter'.
[2] 'Distribution' is used synonymously with 'probability density function' in the following, p. 26.

particular states. Sites for which the local degree of freedom is below the threshold are said to be **stable**, those for which the local degree of freedom is above the threshold are called **active** or **unstable** and those which become unstable once charged are sometimes called **susceptible**.

The models are subject to an **external drive** or **driving**, which changes or **charges** the local variables, continuously or discretely, either everywhere or at one particular point. The former is often called **uniform driving** or **homogeneous driving** (more generally sometimes **global driving**) in particular when the dynamical variable is continuous, whereas no general terminology is established for the latter. Some authors refer to it as **local driving**, some call it **point driving** or **boundary driving** when it takes place at the boundary of the lattice. When the driving takes place with uniform probability throughout the lattice it is often called **(stochastic or random) bulk driving**, in particular when the driving is discrete. The amount of charge transferred from the external source is usually chosen to be small and can be fixed or random. If it involves only one site (or, for that matter, a few sites), the driving position can be fixed or chosen at random. **Stochastic driving** and **deterministic driving** thus both require further specification as to whether that applies to position and/or amount. Finally, if the driving does *not* respect the separation of time scales (see below), it is normally referred to as **continuous driving**.

The external driving plays two different rôles, which are distinguished most clearly when the charges transferred are not conserved under the relaxation described below. On the one hand, the external drive acts as a supply or a **loading** mechanism, which allows the system to respond strongly and very sensitively to an external perturbation. On the other hand, the external driving acts as such a perturbation, **triggering** a potentially very large response. In conserved models like the BTW, MANNA and OSLO Models, but also in the non-conserved OFC Model, the external driving does both loading and triggering. Otherwise, in most non-conserved models like the DROSSEL–SCHWABL Forest Fire Model the two are distinct, whereas in the BS Model loading does not exist as such, because there is nothing to be loaded and there is no quantity, certainly not a conserved one, being transported in response to the triggering event.

Once the external charges trigger the threshold, interaction occurs in the form of **toppling**, which reduces or, more generally, changes the local variable but in turn can lead to the threshold being triggered at neighbouring sites provided they are susceptible. This **relaxation process** defined by the **microscopic dynamics** follows a set of **update rules**, which specifies how degrees of freedom are updated as they interact. Analytical approaches to many models start with an attempt to capture these rules in a mathematical formalism, which can be analysed using established tools of statistical mechanics. These rules can be either **deterministic** or **stochastic**. The most common form of a stochastic relaxation is that interacting sites are picked at random among nearest neighbours. Locally and temporarily the relaxation is therefore **anisotropic**, even when it is on average (in time, space and/or across an ensemble) isotropic. Other forms of stochastic relaxation involve the amount of charge transferred or random values of the degree of freedom at updated sites. The totality of such interaction or **relaxation events** is called an **avalanche**. Avalanching is the archetypal relaxation mechanism in SOC models and is often considered as the signature of **metastability**, as small perturbations due to the external driving can lead to catastrophic responses involving the entire system.

An SOC model is often described as being driven to the brink of stability (also Sec. 1.3.3). Some authors distinguish **stable states**, which respond to external perturbations with small, local changes, **metastable states**,[3] which respond with avalanches of possibly system spanning size, and **unstable states**, which are still changing under the dynamics. Sometimes the distinction between stable and metastable is dropped and both are subsumed under stable. To differentiate the two, the stable states might be labelled as **transient** and the metastable ones as **recurrent** (also Sec. 4.2.1 and Sec. 8.3).

Being so enormously **susceptible**, the system might be considered as being in a **critical state**. Avalanches can be tallied and analysed for their size, duration, the number of sites involved etc. The resulting histograms can be probed for **power laws**, the hallmark of (full) scale invariance. Observables are discussed further in Sec. 1.3.

Any scaling observed in SOC models is usually **finite size scaling**, since the finiteness of the lattice is supposed to be the *only* scale that controls the statistics of the observables. If there is another tunable scale dominating and cutting off the statistical features of avalanches, then the system apparently requires explicit tuning and therefore cannot be called 'self-organised critical'. However, some widely accepted SOC models, e.g. the DS-FFM (Sec. 5.2) possess such a second scale, even when it diverges in some 'trivial' limiting procedure. There is widespread confusion as to what amounts to a second scale and what its consequences are. Generally, *competing* scales are a necessary condition for non-trivial scaling, which can occur only in the presence of dimensionless quantities (Sec. 2.3, p. 48 and Sec. 8.1.3, in particular p. 257). In ordinary critical phenomena, such a second scale needs to be tuned to its particular critical value. If, however, a second scale *dominates* the behaviour of a system over its finite size, it generally is not self-organised critical. The competing scales ultimately succumb to the dominant scale and play a (reduced) rôle in only a finite range of the observable.

While an avalanche is running, the external driving is stopped, known as the **separation of time scales** of driving and relaxation and generally regarded as the key *cause* of SOC. Some authors refer to it as **slow drive**, alluding also to the smallness of the perturbations caused by the external driving. Separation of time scales is achieved as long as the system is driven *slowly enough*, but there is no lower limit, i.e. an SOC system can be driven more slowly without changing its statistical properties. Separation of time scales is akin to the thermodynamic limit, in that it does not require tuning of a control parameter to a particular value that could provide a characteristic scale. Nevertheless, some regard the separation of time scales as a form of tuning.

The time scale on which an avalanche is resolved into individual events of interacting sites governed by the microscopic dynamics is the **microscopic time scale**. As no external charges arrive during an avalanche, these triggering events can be regarded as infinitely far apart on the microscopic time scale. On the **macroscopic time scale**, on the other hand, the external drive has finite (Poissonian) frequency and avalanches are instantaneous, i.e.

[3] Bak *et al.* (1987) called a metastable state (locally) **minimally stable**.

collapse into a point. Such a **complete separation of time scales** where one explodes to infinity from the point of view of the other, which in turn implodes into a point, can often be realised exactly in a computer implementation of the model, simply by not attempting to trigger a new avalanche, while one is running. This way, the driving frequency on the microscopic time scale can be regarded as anything between 0 and the average frequency, i.e. the inverse of the average duration.

To measure these frequencies and other temporal observables, a **microscopic time** needs to be defined explicitly, which is very often not fixed in the rules of the model. In the case of **parallel updating**, sites due for an update are ordered in generations and each generation is dealt with separately, the current generation giving rise to the next. In this case, the microscopic time simply counts the generations and naturally advances by one unit for each parallel update. In the case of **(random) sequential update**, a site is picked at random from all those due to be updated, of which there are, say, N, and time advances by $1/N$, corresponding to the average time spent on each site in a parallel update. If sites are updated as if each were subject to a local **Poisson process**, waiting times between updates can be drawn at random and added to the microscopic time (Sec. A.6.3). Most models remain well defined if the updating scheme is changed, and many models that were originally defined with parallel update are more elegantly defined with random sequential or Poissonian update (in particular the MANNA Model, Sec. 6.1, and OSLO Model, Sec. 6.3), which now is very widely used.

Some authors regard the separation of time scales as a form of global supervision or interaction (Dickman, Vespignani, and Zapperi, 1998) and 'fire the babysitter' (Dickman *et al.*, 2000) by considering a driving frequency that is finite but asymptotically vanishing on the *microscopic time scale*. In a finite system, avalanches (normally) have finite duration, so that a maximum frequency can be found below which the separation of time scales is realised without global supervision, possibly accepting its occasional but very rare violation. This maximum frequency is essentially the inverse of the characteristic duration and diverges with system size much faster than the average duration.

Many models obey an **Abelian symmetry** which is often understood as an invariance of the microscopic dynamics under change of the updating order (Sec. 8.3). Consequently, such models are not uniquely defined on the microscopic time scale and temporal observables might differ in different implementations. **Conservation** during updates of interacting sites *in the bulk* (known as **bulk conservation** or local conservation) is a second important symmetry, which was considered as a necessary ingredient of SOC models very early, but whose rôle was questioned with the arrival of non-conservative models. The quantity conserved is normally the totality of the local dynamical variable, such as the total of the energy or force. Given the external drive, **dissipation** is necessary for a **stationary state** to exist in the presence of bulk conservation. This is often realised by **boundary dissipation** at **open boundaries**, i.e. loss of the otherwise conserved quantity when a boundary site topples. Bulk conservation in conjunction with boundary dissipation leads to **transport** forcing a current through the system, even when it is **isotropic**. If **periodic boundary conditions** apply, for example in order to restore translational variance in a finite lattice, **bulk dissipation** can be implemented explicitly and conservation therefore destroyed.

Many **anisotropic** systems, which have a preferred direction at relaxation, can be dealt with analytically. In **totally asymmetric** models charges occur in only one direction, so that a site that charges another one does not become charged in return when the charged site topples. This effectively suppresses correlations and usually prevents a site from performing **multiple topplings**. Such **directed models** are often exactly solvable.

As mentioned above, SOC was intended to be more than generic scale invariance. To establish the link to ordinary critical phenomena at a transition between phases with different symmetries, in some systems a control parameter can be identified, often the energy, force or particle density (Sornette, 1994, p. 210):

> SOC requires that, as a function of a tunable control parameter, one has a phase transition at some critical point, and that the dynamics of the system brings this parameter automatically to its critical point without external fine-tuning.

For example, many directed models can be mapped to random walks, eliminating initial bias (drift) in a process of self-organisation. If it were not for that process or if the system were placed at criticality *by definition* there would be no point talking about *self-organised* criticality.

In summary, *the most basic design elements of an SOC model are: many interacting degrees of freedom, a local energy or force, a slow external drive and thresholds triggering a fast internal relaxation mechanism (separation of time scales), giving rise to avalanching.* As Jensen (1998) succinctly sums up: SDIDT – 'slowly driven, interaction-dominated threshold systems'. Ideally, an underlying phase transition can be identified which has a temperature-like control parameter tuned to the critical value by the dynamics of the system. Whether or not they exhibit (self-organised) criticality is a matter of observables, to be discussed in the next section.

1.3 Basic observables and observations

The characterisation of avalanches, by size (mass), duration, area covered and radius of gyration, was at the centre of SOC from the very beginning. If avalanches were compiled from (almost) independent patches of toppling sites, their distribution would tend to a Gaussian. Their (non-trivial) power law distribution therefore signals underlying spatiotemporal correlations (Sec. 2.1.3, Sec. 2.3), the direct measurement of which, unfortunately, is technically difficult (and rather unpopular). The distribution also determines the power spectrum, the scaling of which was used to characterise models especially in the early years of SOC. On a more immediate level, power laws do not allow for the definition of a characteristic scale *from within*; to half the probability of a certain event size, it has to be *multiplied* by a *dimensionless* constant. If the distribution is, say, exponential, a *dimensionful* constant, which allows the definition of a characteristic scale, has to be *added* (also p. 355).

The **size s of an avalanche**, sometimes also called the 'mass', is usually defined as the **number of topplings** that occur in the system between an external charge and complete

quiescence, i.e. when all activity has ceased. If the external drive does not lead to an avalanche, its size is defined as $s = 0$. Especially in analytic approaches, it can be advantageous to define the avalanche size as the **number of charges** received throughout the system, including the external drive, so that the smallest possible avalanche size is greater than 0. Depending on the details of the model, the number of topplings and the number of charges have (asymptotically in large avalanches) a fixed ratio, which is complicated by the convention of whether or not to discount the initial charge (external drive) and by dissipation at boundary sites or in the bulk.

Kadanoff, Nagel, Wu, and Zhou (1989) introduced a different measure for the avalanche size, called the **drop number** as opposed to the **flip number** introduced above. The drop number counts the number of particles, energy or force units, that are dissipated at the boundary. Experimentally easier to capture, it was identified early to be somewhat problematic (Jensen, Christensen, and Fogedby, 1989) and has since fallen out of common usage.

The **duration** T of an avalanche is the microscopic time span from the external charge by the driving to complete quiescence. Characteristics of the avalanche duration are riddled with ambiguity when the microscopic time is not defined explicitly. The number of distinct sites toppling (sometimes sites charged) in an avalanche is its **area**, A, and the average distance between every distinct pair of sites toppling during the course of the avalanche is the **radius of gyration**, r. The latter has various alternative definitions, which are borrowed from percolation theory (Stauffer and Aharony, 1994) and polymer science. Of the four features, the avalanche size is by far the most studied, followed by the duration. Much rarer is the analysis of the avalanche area and rarer still analysis of the radius of gyration.

1.3.1 Simple scaling

Avalanche features are typically either tallied into histograms or averaged directly in moments. Measured in an experiment or computer implementation of an SOC model, they can only **estimate** characteristics of the full, 'true' population average. For example, the histogram of the avalanche sizes estimates their **probability density function** (PDF), denoted[4] by $\mathcal{P}^{(s)}(s\,L)$. Such PDFs and derived quantities are subject to a **finite size scaling analysis**, as commonly used for ordinary critical phenomena (Barber, 1983; Privman and Fisher, 1984; Cardy, 1988).

Under the **finite size scaling** (FSS) **hypothesis** the histogram of the avalanche size s is expected to follow **simple scaling** asymptotically for $s \gg s_\mathbf{0}$

$$\mathcal{P}^{(s)}(s; L) = a^{(s)} s^{-\tau} \mathcal{G}^{(s)}(s/s_\mathbf{c}(L)) \quad \text{with} \quad s_\mathbf{c}(L) = b^{(s)} L^{\mathsf{D}}. \tag{1.1}$$

The two **amplitudes** $a^{(s)}$ and $b^{(s)}$ are non-universal **metric factors**, $s_\mathbf{c}$ is the **upper cutoff** or **characteristic avalanche size** and $s_\mathbf{0}$ is the fixed **lower cutoff**. Simple scaling is an asymptote, which approximates the observed histogram increasingly well with increasing ratio $s/s_\mathbf{0}$. Below the constant lower cutoff $s_\mathbf{0}$ the histogram follows a non-universal function, which often depends on the details of the implementation. As opposed to the lower cutoff,

[4] Henceforth, a superscript indicates which observable a particular quantity describes.