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978-0-521-85331-6 — Principles of Astrophysical Fluid Dynamics
Cathie Clarke , Bob Carswell
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Chapter 1

Introduction to concepts

Stated most simply, fluids are ‘things that flow’. This definition distinguishes between liquids and gases (both fluids) and solids, where the atoms are held more or less rigidly in some form of lattice. Of course, it is always possible to think of substances whose status is ambiguous in this regard, such as those, normally regarded as solids, which exhibit ‘creep’ over sufficiently long timescales (glass would fall into this category). Such borderline cases do not undermine the fact that the vast majority of substances can be readily classified as fluid or not. If they *are* fluids, then it is important to understand the general problem of how they flow, and under what circumstances they attain equilibrium (i.e. do not flow). These issues, in an astronomical context, form the subject of this book.

There is also a more subtle point about the sorts of systems that can be described as fluids. Although fluids are always in practice composed of particles at a microscopic level, the equations of hydrodynamics treat the fluid as a continuous medium with well-defined macroscopic properties (e.g. pressure or density) at each point. Such a description therefore presupposes that we are dealing with such large numbers of particles locally that it is meaningful to average their properties rather than following individual particle trajectories. In a similar vein, we may also, for example, treat the dynamics of stars in galaxies as a form of fluid dynamical problem: in this case the ‘particles’ are stars rather than atoms or molecules but the same principles may be used to determine the mean properties of the stars (such as velocity or density) in each region.

In this book, however, we will mainly be concerned with conventional fluids, i.e. liquids and gases. In fact, since the liquid state is hardly encountered apart from in the high pressure environments

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of planetary surfaces and interiors, our focus will very much be on the gas phase (although some of these gases, such as the degenerate gases that compose neutron stars and white dwarfs, bear little resemblance at a microscopic level to conventional gases under laboratory conditions). However, the key property of all gases, as opposed to liquids, is that they are far more *compressible*. Although in many terrestrial applications involving subsonic flows, even gases behave approximately incompressibly, this is not the case in astronomical contexts where flows are frequently accelerated (often by gravity) to high Mach number. This book is therefore not able to make the simplifying assumption, often introduced at an early stage of standard texts on terrestrial fluid mechanics, that the flow is incompressible. Likewise we cannot assume that the battery of techniques for the solution of incompressible flows can be simply generalised to the present case.

1.1 Fluids in the Universe

The baryonic matter in the Galaxy (i.e. conventional matter composed of protons and neutrons) is divided between stars and distributed gas roughly in the ratio 5:1. For the Universe as a whole the ratio is uncertain, but the gas fraction is considerably higher.

Stars are gaseous bodies (mainly hydrogen and helium) with temperatures that range between millions of kelvin in their centres, where nuclear reactions occur, and thousands of degrees at the surface. An easily remembered property of the Sun is that its *mean* density is the same as that of water, but this statistic does not convey its strong internal density stratification (the density at the centre exceeds that at the photosphere – visible surface of the Sun – by 11 orders of magnitude). For some purposes, the interior of stars may be regarded as static, i.e. in a state of force balance between gravity and outwardly directed pressure gradients. In practice, the gas in many stars is subject to internal motions such as convection currents and low amplitude internal oscillations (acoustic modes, see Figure 1.1). Above the photosphere, the gas density falls with increasing height, and the temperature rises, attaining 30 000 K in the so-called chromospheric region where many stellar emission lines originate. At larger distances still, the gas may be magnetically heated to temperatures of around 10^6 K, this coronal region being a strong source of X-rays. We however caution that the low densities in these latter regions mean that a fluid dynamical treatment is not necessarily appropriate (see Section 1.2).

The other main fluid component in the Universe, the distributed gas in the interstellar medium (ISM) and intergalactic medium (IGM), is much more diverse in its properties. For example, the mean density

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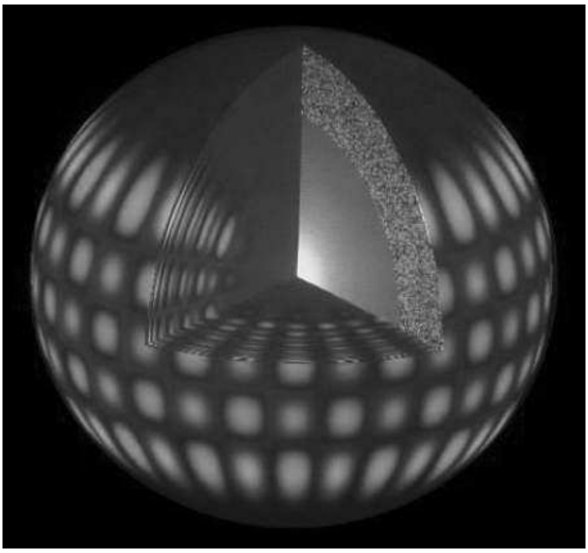

Fig. 1.1. A cut-away illustration showing a spherical harmonic mode of oscillation for acoustic waves in the Sun. (Illustration from Global Oscillation Network Group/National Solar Observatory/AURA/NSF)

of gas in the Milky Way is the easily remembered 1 particle per cubic centimetre, or a million per cubic metre (extraordinarily dilute compared with 2.7×10^{25} particles per cubic metre of gas at standard terrestrial pressure and temperature). This figure however averages over a rich multi-phase medium, comprising warm atomic gas (at $\sim 10^4$ K), a hot phase (at 10^6 K) heated mainly by supernova explosions and a cold molecular phase, which may be as cool as 10 K if well shielded from radiation from bright stars. The density contrasts between these phases are extreme, from ~ 1000 particles per cubic metre for the hot phase to 10^5–10^6 particles per cubic metre for the warm, atomic phase to $\sim 10^8$ particles per cubic metre as a mean for molecular clouds; the densest cores within these clouds have densities in excess of 10^{13} particles per cubic metre. Outside galaxies the densities can be considerably lower, with large regions containing $\lesssim 1$ particle per cubic metre.

Although stars, the ISM and IGM together constitute the bulk of the fluids in the Universe, there are a number of other examples of fluids of astrophysical interest. These include stellar winds, jets and accretion discs on a wide range of scales. Nor should it be forgotten that an important category of stars – the white dwarfs and neutron stars – are also fluid, though with an equation of state – relation between pressure, density and temperature – that is quite different from conventional gases under laboratory conditions. Similarly, the internal structure of the giant planets may be determined as a fluid dynamical problem, although here there are considerable uncertainties

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surrounding the correct equation of state for hydrogen under extreme conditions of density and pressure in the relevant temperature range.

The above thumbnail sketch has stressed the diversity of fluids in the Universe and has perhaps suggested that their study will involve consideration of much complex microphysics. One of the beauties of fluid dynamics is however the way that the microphysical complexity is all contained in a single parameter – the equation of state. Once armed with an equation of state, the fluid dynamicist can simply insert this into the general equations of fluid dynamics. This is not to suggest that the outcome of this exercise is necessarily simple, but there is a pleasing generality that runs through the subject. We will, for example, be able to deduce the structure of white dwarfs and neutron stars as readily as of stars composed of conventional gas, simply because they each have well-defined equations of state and in each case we can consider the stars to be in a state of equilibrium. However, this discussion anticipates much of the contents of later chapters. Before we can embark on deriving the fluid equations, it is necessary that we now introduce some important fluid dynamical concepts.

1.2 The concept of a ‘fluid element’

The fluid equations are based on the concept of a ‘*fluid element*’, i.e. a region over which we can define local variables (such as density, temperature, etc.). The size of the region is assumed to be such that:

(i) It is small enough that we can ignore systematic variations across it for any variable we are interested in, i.e. the region size $\ell_{\rm region}$ is much smaller than a scale length for change of any relevant variable q (where a scale length is the scale over which q varies by order unity). So

$$\ell_{\rm region} \ll \ell_{\rm scale} \sim \frac{q}{|\boldsymbol{\nabla} q|}. \tag{1.1}$$

(ii) It is large enough that the element contains sufficient particles for us to ignore fluctuations due to the finite number of particles (‘discreteness noise’). Thus if the number density of particles per unit volume is n, we require that

$$n\ell^3_{\rm region} \gg 1. \tag{1.2}$$

The above two criteria must be met by *any* system that is describable as a fluid. In addition, *collisional* fluids must satisfy the following criterion:

(iii) The fluid element is large enough that the constituent particles 'know' about local conditions through colliding with each other, so if the mean free path is λ, we require that

$$\ell_{\text{region}} \gg \lambda. \tag{1.3}$$

What do we mean by a collisional fluid? The essential point is that if the particles in a fluid interact with each other (which does not necessarily imply that they physically collide), then they will attain a distribution of, say, particle speed that maximises the entropy of the system at that temperature. (For this discussion we can simply regard the entropy of a system as being a measure of the number of microscopically distinct configurations that correspond to a given macroscopic property of the fluid locally.) Therefore a collisional fluid at a given temperature and density will have a well-defined distribution of particle speeds in the local rest frame, and hence a corresponding pressure. Thus one can derive an equation of state (relationship between density, temperature and pressure) for a collisional fluid. Almost all the fluids we consider in this book, be they ideal gases or degenerate, are collisional fluids with corresponding equations of state. However, as noted above, we can also consider systems of orbiting stars in a galaxy, or grains in Saturn's rings, as being fluids, even though the particles now do not interact sufficiently frequently to satisfy criterion (1.3) above. In this case, the distribution of particle speeds locally does not correspond to an entropy maximum but instead depends on the initial distribution of particle speeds. The fact that one cannot write down an equation of state for collisionless systems means that it is a hard problem, whose solution is non-unique, to find, for example, the structure of stellar orbits in a galaxy in equilibrium. In this book we will avoid this difficulty by considering collisional fluids (i.e. conventional liquids and gases).

It should be stressed that a fluid element is purely a conceptual entity – ℓ_{region} does not enter into the fluid equations. However, if a system is to be described by the fluid equations, its properties must be such that there are values of ℓ_{region} that simultaneously satisfy the conditions above.

1.3 Formulation of the fluid equations

There are two (sensible) approaches to formulating the equations for mass density, momentum and energy in a fluid:

(i) Eulerian description
Consider a small volume at a fixed spatial position. The fluid flows through the volume with physical variables specified as functions of time and the (fixed) position of the small volume: density $\rho = \rho(\mathbf{r}, t)$,

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temperature $T = T(\mathbf{r}, t)$, etc. The change of any measurable quantity as a function of time at that position is $\partial/\partial t$ of the quantity, evaluated at the *fixed position*.

(ii) Lagrangian description
In this approach one chooses a particular fluid element and examines the change in variables (density, temperature, etc.) in that particular element. So the (spatial) reference system is comoving with the fluid. Thus one might examine the behaviour of $\rho = \rho(\mathbf{a}, t)$, where $\mathbf{a}$ is a label for a particular fluid element – which might be the coordinates at a chosen initial time, for example. The time derivative is now a partial one at a fixed $\mathbf{a}$ (i.e. for a fixed bit of the fluid), and the rate of change with respect to time for a *fixed element* is denoted $\mathrm{D}/\mathrm{D}t$.

In the Lagrangian description, position is not an independent variable but instead $\mathbf{r} = \mathbf{r}(\mathbf{a}, t)$. The Lagrangian description refers to the world as seen by an observer riding on a fluid element (e.g. a river viewed from a boat adrift on it); the Eulerian one refers to the world as seen at a fixed spatial position (e.g. a river viewed from the bank).

The Lagrangian approach is useful if the path of an individual element is important, e.g. when a particular element has some property which distinguishes it from all the others. Usually this is not the case; however, one can think of particular instances where it is important in astronomy (e.g. tracing the trajectory of a parcel of gas that has been enriched by heavy elements as it is ejected into the interstellar medium by a supernova). The Eulerian approach is usually more useful if the motion of individual fluid elements is not of interest. It is particularly useful for *steady* flows, i.e. those where the quantities at a given position do not change. Then $\partial/\partial t$ of all variables $= 0$ everywhere. Steady flows have no special properties in Lagrangian descriptions since in a steady flow an individual element still changes its properties in general as it goes from place to place.

This conceptual split between the Eulerian and Lagrangian formulations translates into two entirely distinct ways of simulating fluid dynamical problems on a computer. Eulerian codes set up a grid of fixed boxes and compute the changes of all variables in each box as the flow evolves. Lagrangian codes instead set up an ensemble of particles which represent fluid elements and follow the trajectories of the particles in the flow. There are advantages and disadvantages to each approach which are being much discussed at present in relation to simulating the formation of stars and galaxies, two highly topical problems in astrophysical fluid dynamics. Eulerian codes in their simplest form have the disadvantage that you have to decide at the beginning of the calculation how you are going to distribute your grid cells, i.e. in what regions of the flow do you want fine gridding (high

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resolution). However, as the flow evolves you might need high resolution in a region that you would not have predicted at the outset of the calculation, unless your problem has very regular symmetry. The Lagrangian approach circumvents this difficulty since high resolution will automatically be achieved in high density regions of the flow where lots of particles end up. However, the problem with using particles to model a continuous fluid is then how to compute the density at each point (number of particles divided by some sampling volume is an obvious approach, but then the density calculated might fluctuate unphysically as particles enter and leave the sampling volume). Much work has gone into devising codes that minimise the inherent noisiness of particle based methods, particularly a class of codes known as Smoothed Particle Hydrodynamics (SPH). In the meantime, Eulerian codes are becoming more cunning through developing the capability of reconfiguring the grid automatically during the calculation in order to achieve high resolution where it is needed (Adaptive Mesh Refinement methods: AMR). There is a large and evolving literature on computing methods for fluid problems – see e.g. the Von Karman Institute Lecture Series Monograph: *Computational Fluid Dynamics*.

1.4 Relation between the Eulerian and Lagrangian descriptions

Consider any quantity Q (which may be a scalar or a vector) in a fluid element which is at position $\mathbf{r}$ at time t. At time $t+\delta t$ the element is at $\mathbf{r}+\delta\mathbf{r}$, and then, straight from the definition

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t\to 0}\left[\frac{Q(\mathbf{r}+\delta\mathbf{r}, t+\delta t) - Q(\mathbf{r}, t)}{\delta t}\right]. \tag{1.4}$$

The numerator is

$$\begin{aligned} Q(\mathbf{r}+\delta\mathbf{r}, t+\delta t) - Q(\mathbf{r}, t) = {} & Q(\mathbf{r}, t+\delta t) - Q(\mathbf{r}, t) \\ & + Q(\mathbf{r}+\delta\mathbf{r}, t+\delta t) - Q(\mathbf{r}, t+\delta t) \end{aligned} \tag{1.5}$$

which is, to first order in $\delta\mathbf{r}$, δt,

$$= \frac{\partial Q(\mathbf{r}, t)}{\partial t}\delta t + \delta\mathbf{r}\cdot\boldsymbol{\nabla} Q(\mathbf{r}, t+\delta t), \tag{1.6}$$

and so, expanding the second term,

$$= \frac{\partial Q(\mathbf{r}, t)}{\partial t}\delta t + \delta\mathbf{r}\cdot\left[\boldsymbol{\nabla} Q(\mathbf{r}, t) + \frac{\partial \boldsymbol{\nabla} Q}{\partial t}\delta t\cdots\right]. \tag{1.7}$$

The $\delta\mathbf{r}\cdot\frac{\partial\boldsymbol{\nabla} Q}{\partial t}\delta t$ term is of second order in small quantities, so, in the limit as δt and $\delta r \to 0$,

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \frac{\partial Q}{\partial t} + \mathbf{u} \cdot \nabla Q, \tag{1.8}$$

where **u** is the fluid velocity.

The Lagrangian time derivative has a term due to the rate of change at a fixed location (i.e. the Eulerian time derivative) plus a term due to the fact that the fluid element has moved to a new location where the variable has a different value. The extra term is known as the ‘convective derivative’.

As a reminder, for scalar Q, $\mathbf{u} \cdot \nabla Q$ in Cartesian, spherical polar and cylindrical polar coordinates is

$$\begin{aligned}\mathbf{u} \cdot \nabla Q &= u_x \frac{\partial Q}{\partial x} + u_y \frac{\partial Q}{\partial y} + u_z \frac{\partial Q}{\partial z} \\ &= u_r \frac{\partial Q}{\partial r} + \frac{u_\theta}{r} \frac{\partial Q}{\partial \theta} + \frac{u_\phi}{r \sin\theta} \frac{\partial Q}{\partial \phi} \\ &= u_R \frac{\partial Q}{\partial R} + u_z \frac{\partial Q}{\partial z} + \frac{u_\phi}{R} \frac{\partial Q}{\partial \phi}.\end{aligned}$$

If **Q** is a vector, then in Equation (1.8) $\mathbf{u} \cdot \nabla \mathbf{Q}$ is also a vector, each component of which is $\mathbf{u} \cdot \nabla$ acting on each component of **Q**. So, in Cartesians, where $\mathbf{Q} = (Q_x, Q_y, Q_z)$, we have

$$\begin{aligned}\mathbf{u} \cdot \nabla \mathbf{Q} = \Bigg(& u_x \frac{\partial Q_x}{\partial x} + u_y \frac{\partial Q_x}{\partial y} + u_z \frac{\partial Q_x}{\partial z}, u_x \frac{\partial Q_y}{\partial x} + u_y \frac{\partial Q_y}{\partial y} + u_z \frac{\partial Q_y}{\partial z}, \\ & u_x \frac{\partial Q_z}{\partial x} + u_y \frac{\partial Q_z}{\partial y} + u_z \frac{\partial Q_z}{\partial z} \Bigg).\end{aligned} \tag{1.9}$$

For these expressions in spherical polars, or cylindrical coordinates, see Appendix.

1.5 Kinematical concepts

Kinematics is the study of particle trajectories. It is distinct from dynamics in that kinematics does not concern itself with the origin of particle motions but just analyses various properties of particles moving in a known velocity field $\mathbf{u}(\mathbf{r}, t)$ (i.e. a vector field defined everywhere in Eulerian coordinates).

(i) Streamlines

The defining feature is that the tangent to a streamline at any point gives the direction of the velocity at that point. The tangent to a curve with parameter s is given in Cartesian coordinates by the vector $(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}z}{\mathrm{d}s})$, so streamlines are determined by the following system of equations:

$$\frac{\mathrm{d}x}{u_x} = \frac{\mathrm{d}y}{u_y} = \frac{\mathrm{d}z}{u_z}, \tag{1.10}$$

where the u are evaluated everywhere at the particular time of interest.

In spherical polars the form is slightly different:

$$\frac{\mathrm{d}r}{u_r} = \frac{r\,\mathrm{d}\theta}{u_\theta} = \frac{r\sin\theta\,\mathrm{d}\phi}{u_\phi}, \tag{1.11}$$

just reflecting the fact that we chose to have each component of $\mathbf{u}$ as the velocity projection in the orthogonal coordinate increment directions, $\mathbf{u} = (\frac{\mathrm{d}r}{\mathrm{d}t}, r\frac{\mathrm{d}\theta}{\mathrm{d}t}, r\sin\theta\frac{\mathrm{d}\phi}{\mathrm{d}t})$.

In cylindrical polars:

$$\frac{\mathrm{d}R}{u_R} = \frac{\mathrm{d}z}{u_z} = \frac{R\,\mathrm{d}\phi}{u_\phi}. \tag{1.12}$$

(ii) Particle paths

The paths of individual particles as functions of time satisfy

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{u}(\mathbf{r}, t). \tag{1.13}$$

The constant of integration labels the different particles – for example you might use $\mathbf{r}(t = 0)$ to label a particle. The particle paths follow the streamlines for small times relative to the start time, since the $\mathbf{u}$ may be treated as a constant then, but if the flow is time dependent then the streamlines and the particle paths are not the same.

The equations of particle paths in various coordinate systems are as follows:

Cartesian:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_x, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = u_y, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = u_z.$$

Spherical:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u_r, \qquad r\frac{\mathrm{d}\theta}{\mathrm{d}t} = u_\theta, \qquad r\sin\theta\frac{\mathrm{d}\phi}{\mathrm{d}t} = u_\phi.$$

Cylindrical:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = u_R, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = u_z, \qquad R\frac{\mathrm{d}\phi}{\mathrm{d}t} = u_\phi.$$

(iii) Streaklines

A streakline is the line (at a particular time t) joining the instantaneous positions of all the particles which have ever passed (and will ever

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pass) through a particular point. The way to visualise this is to think of all particles passing through a given point being dyed red as they do so: the streakline is the resulting red line (for example, these are the lines you see in some wind tunnel experiments).

The equations of a streakline involve determining what subset of particles ever pass through a particular point $\mathbf{r}_0$, i.e. for what particle labels $\mathbf{a}$ is $\mathbf{r}(\mathbf{a}, t) = \mathbf{r}_0$ for some value of t? The streakline is then $\mathbf{r}(\mathbf{a}, 0)$, where $\mathbf{a}$ ranges through all the values satisfying the condition, and $\mathbf{r}_0$ labels each streakline.

Note that for *steady* flows streamlines, streaklines and particle paths are the *same*. For a steady flow, $\frac{\partial}{\partial t} = 0$, so every particle passing through a given point follows the same path.

For an example where they are different we need a non-steady flow. Suppose we have a flow where, for $t < 0$, $\mathbf{u} = (1, 0, 0)$, and for $t > 0$ $\mathbf{u} = (0, 1, 0)$. The streamlines are shown in Figure 1.2.

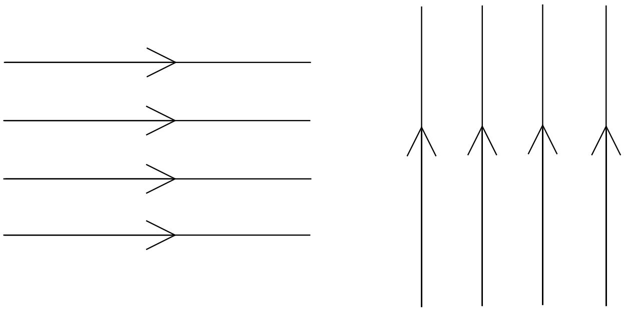

Fig. 1.2. Streamlines for $t < 0$ (left) and $t > 0$ (right).

The particle paths reflect this change in velocity:

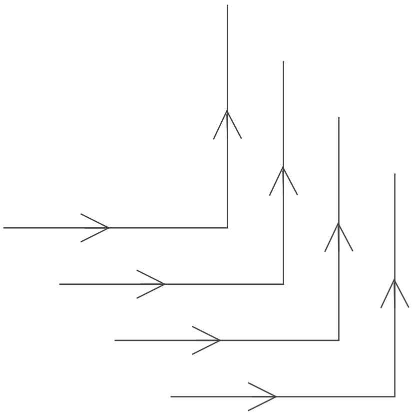

Fig. 1.3. Particle paths, where the change in direction occurs at $t = 0$.