LAGRANGIAN FLUID DYNAMICS

The emergence of observing systems such as acoustically-tracked floats in the deep ocean, and surface drifters navigating by satellite, has seen renewed interest in Lagrangian fluid dynamics.

Starting from the foundations of elementary kinematics and assuming some familiarity of Eulerian fluid dynamics, this book reviews the classical and new exact solutions of the Lagrangian framework, and then addresses the general solvability of the resulting general equations of motion. A unified account of turbulent diffusion and dispersion is offered, with applications among others to plankton patchiness in the ocean.

Designed as a graduate-level text and work of reference, the book provides the first detailed and comprehensive analytical development of the Lagrangian formulation of fluid dynamics, of interest not only to applied mathematicians but also oceanographers, meteorologists, mechanical engineers, astrophysicists and indeed all investigators of the dynamics of fluids.

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To John Joseph Mahony

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Preface

Motivation

Leaves drifting in streams and blowing in the wind belong amongst our root impressions of the natural world. Plumes discharging into streams and pumping from smoke stacks symbolize our impact on that world. Thus it is baffling when as students we discover that fluid dynamics is seemingly exclusively investigated by measuring pressure at fixed points. The manometers in our first fluids laboratories plainly measure total stagnation pressure; the mechanical flow meters less obviously strike a dynamical balance between the torque of the partial stagnation pressure on the turbine blades and the torque of friction in the turbine bearings. Our hands and faces do feel the rush of a stream or the sweep of the wind, but these are brute sensations in comparison to the incisive information processing at work when our eyes follows a flow marker.

This is a book about the role of kinematics in fluid dynamics. The most revealing mathematical framework for developing kinematics is the Lagrangian formulation, long ago discarded for being unwieldy compared to the Eulerian formulation (Tokaty, 1971). Yet the discarded unwieldiness owes precisely to the richness of the kinematical information. This book might have been written any time in the twentieth century; the motivation now is the emergence of Lagrangian observing technology. The emergence is of course a reemergence; meteorologists have been routinely tracking weather balloons with theodolites since the nineteenth century. However, visual tracking and short transmitter life limit these data to being little more than local or Eulerian measurements of wind velocity and thermodynamic conditions. Radar, acoustics, satellite relays and satellite-based navigation changed all that in the late twentieth century. High-altitude balloons were tracked by satellites for days during the First GARP Global Experiment (WMO, 1977). Floats deep in the ocean are now tracked, effectively continuously, for months and even years with onboard hydrophones and moored arrays of pingers.

Aims

The Lagrangian formulation of fluid dynamics is not likely to replace the Eulerian formulation. Such is especially the case in computational fluid dynamics, although hybrid techniques are gaining ascendency. Rather, the Lagrangian formulation complements the Eulerian. Hence this book is not intended as a first course in fluid dynamics, and readers are assumed to have studied Eulerian fluid dynamics at least at the level of an introductory course having a scientific rather than engineering bent (Batchelor, 1973). Advanced calculus (Apostol, 1957) and Cartesian tensors (Jeffreys, 1931) are of necessity used extensively. The treatment of turbulence here assumes considerable preliminary familiarity with empirical, dimensional and statistical aspects (Lumley and Panofsky, 1964; Tennekes and Lumley, 1972). The purpose of the treatment developed here, indeed the purpose of the book as a whole, is to reveal the unifying power of the Lagrangian formulation for one of the great problems in physics. The further purpose is the drawing of a broader perspective for the analysis of the important new environmental data being collected with the emerging Lagrangian technologies.

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The development of Lagrangian fluid dynamics falls naturally into major parts. Part I is concerned with the essence of the Lagrangian formulation, beginning with the kinematics of particles and the introduction of a sufficiently powerful notation for particle kinematics. The reader is advised against skipping lightly through this seemingly prosaic material. It quantifies the concept of conservation of particle identity, which is perhaps *the* intrinsically Lagrangian concept. The concept is captured by the *labeling theorem* of Kraichnan (1965), also known as *Lin's identity* (Lin, 1963). A striking corollary of this theorem is an exact expression for a generalized Lagrangian drift in a laminar flow and in each realization of a turbulent flow. Approximate drift formulae have long been the subject of speculation: here is the actuality. While the first candidate for a dependent variable in Lagrangian fluid dynamics is the particle path, the more readily observed structures are streamlines in a wind tunnel or towing tank, and streaklines downstream of sources such as discharge pipes and smoke stacks. In anticipation of the complexity of such flow, the introduction of statistical quantification is essential. The rudiments are found here; comprehensive treatments may be found elsewhere, for example Monin and Yaglom (1971, 1975). A few generalities may be made for single particle and particle pair statistics in homogeneous turbulence, that is, in turbulent flow which is statistically uniform in space. The problem of relating Eulerian and Lagrangian statistics is shown to be formally solved with functional or 'path' integrals.

The Lagrangian developments of dynamical principles into conservation laws for mass, momentum and energy should be familiar since they are found in most purportedly Eulerian texts. This familiarity underscores the greater directness and clarity of the Lagrangian formulation of Newtonion dynamics for fluids. The momentum equation in particular involves particle accelerations; these are second-order partial derivatives of particle position with respect to time elapsed since identification or release. Both Cauchy and Weber realized (Lamb, 1932) that one integration in time is immediately feasible. Pressure is supplanted by another scalar invariant, while Cauchy's vector invariant usurps vorticity. The Cauchy invariant reveals that neither the particle path nor the particle velocity is the intrinsic dependent variable in fluid dynamics; rather it is the Jacobi matrix or strain matrix of partial derivatives of position with respect to initial position, or with respect to whatever dependent variable identifies or labels the particle. Unlike particle position, the Jacobi matrix is invariant with respect to Galilean transformations of space. To split the hair, both the Eulerian equations of motion and the original Lagrangian equations of motion are Galilean invariant; it is their respective dependent variables of velocity and position which are not. Two Russian hydrodynamicists have recently pointed out that there is a matrix notation for the strain-based development (Yakubovich and Zenkovich, 2001). While this compact notation appears to offer no advantages for numerical computations, it has enabled its proponents to generalize the rotational wave of Gerstner (Lamb, 1932) to a new class of vortices.

Lagrangian fluid dynamics can be expressed as a variational principle; the invariance of the Lagrange density with respect to changes of particle labels leads to the fundamentally important conservation laws for Ertel's potential vorticity. The laws are derived here with the care that is owed, to the extent that the widely claimed naturality for the variational approach is not so compelling.

Lagrangian variables, both dependent and independent, need coordinates. All the coordinate options in the Eulerian formulation are available. The detailed forms for the Lagrangian equations in various coordinates suggest symmetries which are global in space: the familiar transformations representing rotations, which leave the equations invariant in form, are independent of position but may depend upon time. Of particular interest to meteorologists and oceanographers is the form of the Lagrangian equations in a uniformly rotating reference frame.

Real fluids are characterized by the constitutive relations between stress and strain. Newtonian fluids are defined by a linear relationship between the local stress tensor and the local rate of strain tensor. The locality is essentially Eulerian in nature. It is most simply expressed with Eulerian variables, and is particularly awkward in Lagrangian variables. Yet, again, the appearance of strain components, some of which may be rapidly growing, makes manifest the tendency for intensification of viscous stresses by differential particle motion. The locality of the Newtonian stress tensor is an expression of loss of memory, while the strength of the Lagrangian formulation is memory expressed as the retention of fluid particle identity. Which is the closer to reality: loss of memory or memory retention? While the Newtonian constitutive relation is of undisputed practical value, it is not so much a fundamental physical law as a "phenomenological" law, to use the language of Prigogine (1980). In other words, a fluid continuum is an abstraction, an unnatural artifice. Real air and real water consist of assemblies of molecules, obeying the fundamental laws of conservation of mass, momentum and energy. Viscous stresses are caused by nonequilibrium distributions of molecular velocity, as shown by the Chapman–Enskog deduction of the Navier–Stokes equations from Boltzmann's equation (which deduction must surely qualify as the "grand unified field theory" of the early twentieth century, and in fact of much of the world which really matters to us; see Chapman and Cowling, 1970). Alas, Boltzmann's *stosszahlansatz* is an admission of loss of memory at the molecular level (Thompson, 1988), so Lagrangian memory retention would seem to be in vain. Yet a complete topological description of the motion history of the macroscopic medium – the fluid continuum – demands a formulation in which memory retention is intrinsic. The crisis was created not by the development of the Lagrangian formulation, but initially by Boltzmann having randomizing Liouville's detailed microscopic description of molecular motion, and subsequently by Chapman and Enskog having taken moments of Boltzmann's distributional description.

Having declared that the fully Lagrangian formulation of fluid dynamics appears to offer no great numerical computational advantage, it would be desirable to be able to offer a great range of analytical Lagrangian solutions. Alas, there are only a few and these are also presented in Part II. Then again, there are about as few analytical Eulerian solutions, and strictly Eulerian numerical methods are being overtaken by semi-Lagrangian methods. It is curious that some problems admit explicit solutions in one formulation but not in the other. Irrotational flow past a circular cylinder admits an explicit Eulerian solution, but the Lagrangian solution is not explicit. The latter is presented here simply to make the point. On the other hand, there is no explicit Eulerian solution for the Gerstner wave. Nor are there for its generalizations, the Ptolemaic vortices of Yakubovich and Zenkovich (2001). There are analytical Lagrangian solutions for planar flows of real fluids, typically near flat plates. The Navier–Stokes equations for steady, incompressible viscous flow in a flat-plate boundary layer were simplified by Prandtl (Schlichting, 1960); as pointed out by Blasius (Schlichting, 1960), Prandtl's equations admit a single similarity variable, and the resulting nonlinear ordinary differential equation may be solved numerically. It is shown here that the Lagrangian form of Prandtl's equations admit two similarity variables, one of which includes time, leading to a pair of partial differential equations.

The Lagrangian formulation may be derived from the Eulerian by a transformation of variables, but the transformation is flow dependent. The two formulations are therefore sufficiently different from a mathematical point of view that the general solvability of the Lagrangian must be addressed. Indeed, the increasing interest in numerical Lagrangian fluid dynamics motivates the question: is the computer really computing a flow?

It has long been recognized that the Lagrangian formulation is natural for the analysis of conserved passive tracers. The formulation for diffusing tracers is greatly complicated by the appearance of the Jacobi matrix, but assuming the strain components are uniform in space permits an analytical solution. The assumption turns out to be a valid approximation for turbulent diffusion on a broad range of scales; the Lagrangian solution developed throughout Part III provides a unifying theoretical development of the many subranges of homogeneous turbulence, and for diffusion of concentration gradients. The results go beyond mere dimensional consistency or similarity, correctly generating functional forms in subranges where alternative forms coexist, some of which are dimensionally consistent but wrong. Relative dispersion is shown to interact with spatially nonuniform plankton growth rates to destroy spatial patchiness in the plankton concentration. Part III, which offers a coherent and strictly Lagrangian presentation of turbulent diffusion ranging from microscales in liquid mercury to planetary scales in the stratosphere, is a completely reworked, reargued and augmented edition of an essay which first appeared in *Reviews of Geophysics* (Bennett, 1987).

No coherent presentation of Lagrangian fluid dynamics appears to have been offered prior to this book, but there are comprehensive accounts of a number of hybrid formulations. Their being both hybrid formulations and well described elsewhere, there is no need to cram them in here. The Abridged Lagrangian History Direct Interaction Approximation (Kraichnan, 1965; Frisch, 1995) is a perturbative development of stationary, isotropic turbulence. The formulation is indeed hybrid, having both Lagrangian and Eulerian aspects. "ALHDIA" yields the correct self-similar inertial subrange, while an analagous but strictly Eulerian formulation does not. ALHDIA also yields the viscous subrange observed definitively in a tidal channel in British Columbia (Grant, Stewart and Moilliet, 1962); as such the theory is one of the great unsung victories of middle twentieth century physics. The hybdrid Lagrangian formulation by Andrews and McIntyre (1978) permits Reynolds' averaging without loss of operator form. Applied to the atmosphere, the Lagrangian mean formulation facilitates the analysis of beams of internal waves. As repeatedly mentioned here, semi-Lagrangian numerical methods are pervading all of computational fluid dynamics; there are many introductory accounts (e.g., Durran, 1999). Finally come random flight models, which take the form of stochastic differential equations. The models are traditionally if mistakenly described as Lagrangian simulations. After all, they originated in Einstein's theory of the Brownian motion of minute but distinct particles in water (Pais, 1982). Stochastic differential equations are, as far as scientific content is concerned, no more than elegant algorithms for solving the associated diffusion equations, and the approximate closures that lead from the true probabilistic Lagrangian kinematics to the diffusion equations are profoundly suspect. Nevertheless, Rodean (1996) presents a comprehensive treatment of Monte Carlo simulation of turbulent diffusion. Only a very brief outline is included here, with application again to plankton dynamics.

The emerging Lagrangian observing technologies that so much motivate this book are reviewed in Part IV. The brief data survey includes many World Wide Web addresses for sites supporting these technologies, especially oceanographic surface 'drifters' and subsurface 'floats'.

It is shown in Chapter 7 of Part II that simple wave solutions of infinitesimal amplitude may be developed in the Lagrangian framework, just as in the Eulerian framework. Sums of the Eulerian wave solutions have routinely been fitted to real data, but on scales that deny the assumptions upon which the simple wave solutions are based such as an unbounded, uniform and constant medium of propagation. Yet we continue to torture the real atmosphere and real ocean on this Procrustean bed of simple wave expansions. The practice should be deemphasized in favor of inverse modeling, that is, finding fields that simultaneously give good fits to the finite amplitude equations of fluid dynamics in a realistically shaped and realistically stratified ocean basin, and to real ocean data. Both fits should be sought within hypothesized levels of error. By implication a dynamical model should include not just the equations of motion, initial conditions and boundary conditions, but also quantitative estimates of the errors in each of these component. Any failure to fit would most likely indicate an overoptimistically small prior for the dynamical errors, that is, something new would have been learned about ocean dynamics. The Eulerian theory of oceanic and atmospheric inverse modeling may be found elsewhere (Bennett, 1992, 2002; Wunsch, 1996). Emerging methods of Lagrangian inverse modeling and Lagrangian data assimilation in general are introduced in Part IV.

Again, dynamical investigations of fluid motion must move beyond approximate analytical solutions and "forward" numerical integrations of the dynamics, followed by simple comparisons with data. In the preferable inverse calculations, the dynamical constraints need not be satisfied exactly, so the conventional dynamical insights obtained by closely evaluating Eulerian term balances do not apply in general. In any event, the term balance approach is frought with difficulties on planetary scales, since many processes contribute to the balances as a rule and their respective roles in the balances vary substantially over the ocean basin in question. The search for local dynamical insights must be complemented with new and advanced skills at tracking fluid particles, estimating the convergence and divergence of these tracks and assessing the impact of such kinematic processes on the evolving pressure gradients, that is, on the dynamics. Much experience is needed, in order that insights may be drawn from the combination of inverse modeling and the Lagrangian perspective. For instance, Eulerian analysis of deep float data routinely involves unconstrained linear regression, for the estimation of the Eulerian pressure from the float tracks. Yet pressure is not the dynamically appropriate scalar field from the Lagrangian perspective. The ocean analyst should instead estimate the scalar field of Cauchy and Weber, in a manner consistent with Lagrangian kinematics and dynamics. If this book is effective, the next generation of physical oceanographers will be able to do so.

Ulterior motive

It should be evident that considerable amounts of mathematical needlepoint are required for the Lagrangian analysis of fluid dynamics. Today's students have a marvelous facility with computers, even though their manipulative skills are less honed. Equally admirable are the students of an earlier generation who could knock off the Tripos questions in Whittaker and Watson's

Modern Analysis. The author falls between the two generations, yet wishes to provide some opportunities and encouragement to today's students so that they might acquire some of the older masteries.

Corvallis, 2005.

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