

Contents

<i>Preface</i>	<i>page</i>	xiii
1 Basic concepts		1
1.1 Elementary properties of Lie algebras		1
1.2 Representations and modules		5
1.3 Abelian, nilpotent and soluble Lie algebras		7
2 Representations of soluble and nilpotent Lie algebras		11
2.1 Representations of soluble Lie algebras		11
2.2 Representations of nilpotent Lie algebras		14
3 Cartan subalgebras		23
3.1 Existence of Cartan subalgebras		23
3.2 Derivations and automorphisms		25
3.3 Ideas from algebraic geometry		27
3.4 Conjugacy of Cartan subalgebras		33
4 The Cartan decomposition		36
4.1 Some properties of root spaces		36
4.2 The Killing form		39
4.3 The Cartan decomposition of a semisimple Lie algebra		45
4.4 The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$		52
5 The root system and the Weyl group		56
5.1 Positive systems and fundamental systems of roots		56
5.2 The Weyl group		59
5.3 Generators and relations for the Weyl group		65

6	The Cartan matrix and the Dynkin diagram	69
6.1	The Cartan matrix	69
6.2	The Dynkin diagram	72
6.3	Classification of Dynkin diagrams	74
6.4	Classification of Cartan matrices	80
7	The existence and uniqueness theorems	88
7.1	Some properties of structure constants	88
7.2	The uniqueness theorem	93
7.3	Some generators and relations in a simple Lie algebra	96
7.4	The Lie algebras $L(A)$ and $\check{L}(A)$	98
7.5	The existence theorem	105
8	The simple Lie algebras	121
8.1	Lie algebras of type A_l	122
8.2	Lie algebras of type D_l	124
8.3	Lie algebras of type B_l	128
8.4	Lie algebras of type C_l	132
8.5	Lie algebras of type G_2	135
8.6	Lie algebras of type F_4	138
8.7	Lie algebras of types E_6, E_7, E_8	140
8.8	Properties of long and short roots	145
9	Some universal constructions	152
9.1	The universal enveloping algebra	152
9.2	The Poincaré–Birkhoff–Witt basis theorem	155
9.3	Free Lie algebras	160
9.4	Lie algebras defined by generators and relations	163
9.5	Graph automorphisms of simple Lie algebras	165
10	Irreducible modules for semisimple Lie algebras	176
10.1	Verma modules	176
10.2	Finite dimensional irreducible modules	186
10.3	The finite dimensionality criterion	190
11	Further properties of the universal enveloping algebra	201
11.1	Relations between the enveloping algebra and the symmetric algebra	201
11.2	Invariant polynomial functions	207
11.3	The structure of the ring of polynomial invariants	216
11.4	The Killing isomorphisms	222

<i>Contents</i>		ix
11.5	The centre of the enveloping algebra	226
11.6	The Casimir element	238
12	Character and dimension formulae	241
12.1	Characters of L -modules	241
12.2	Characters of Verma modules	244
12.3	Chambers and roots	246
12.4	Composition factors of Verma modules	255
12.5	Weyl's character formula	258
12.6	Complete reducibility	262
13	Fundamental modules for simple Lie algebras	267
13.1	An alternative form of Weyl's dimension formula	267
13.2	Fundamental modules for A_l	268
13.3	Exterior powers of modules	270
13.4	Fundamental modules for B_l and D_l	274
13.5	Clifford algebras and spin modules	281
13.6	Fundamental modules for C_l	292
13.7	Contraction maps	295
13.8	Fundamental modules for exceptional algebras	303
14	Generalised Cartan matrices and Kac–Moody algebras	319
14.1	Realisations of a square matrix	319
14.2	The Lie algebra $\tilde{L}(A)$ associated with a complex matrix	322
14.3	The Kac–Moody algebra $L(A)$	331
15	The classification of generalised Cartan matrices	336
15.1	A trichotomy for indecomposable GCMs	336
15.2	Symmetrisable generalised Cartan matrices	344
15.3	The classification of affine generalised Cartan matrices	351
16	The invariant form, Weyl group and root system	360
16.1	The invariant bilinear form	360
16.2	The Weyl group of a Kac–Moody algebra	371
16.3	The roots of a Kac–Moody algebra	377

17	Kac–Moody algebras of affine type	386
17.1	Properties of the affine Cartan matrix	386
17.2	The roots of an affine Kac–Moody algebra	394
17.3	The Weyl group of an affine Kac–Moody algebra	404
18	Realisations of affine Kac–Moody algebras	416
18.1	Loop algebras and central extensions	416
18.2	Realisations of untwisted affine Kac–Moody algebras	421
18.3	Some graph automorphisms of affine algebras	426
18.4	Realisations of twisted affine algebras	429
19	Some representations of symmetrisable Kac–Moody algebras	452
19.1	The category \mathcal{O} of $L(A)$ -modules	452
19.2	The generalised Casimir operator	459
19.3	Kac’ character formula	466
19.4	Generators and relations for symmetrisable algebras	474
20	Representations of affine Kac–Moody algebras	484
20.1	Macdonald’s identities	484
20.2	Specialisations of Macdonald’s identities	491
20.3	Irreducible modules for affine algebras	494
20.4	The fundamental modules for $L(\tilde{A}_1)$	504
20.5	The basic representation	508
21	Borcherds Lie algebras	519
21.1	Definition and examples of Borcherds algebras	519
21.2	Representations of Borcherds algebras	524
21.3	The Monster Lie algebra	530
	<i>Appendix</i>	540
	Summary pages – explanation	540
	Type A_l	543
	Type B_l	545
	Type C_l	547
	Type D_l	549
	Type E_6	551
	Type E_7	553
	Type E_8	555
	Type F_4	557
	Type G_2	559

<i>Contents</i>		xi
Type \tilde{A}_1		561
Type $\tilde{A}'_1 = {}^2\tilde{A}_2$	(1st description)	563
	(2nd description)	565
Type \tilde{A}_l		567
Type \tilde{B}_l		570
Type $\tilde{B}'_l = {}^2\tilde{A}_{2l-1}$		573
Type \tilde{C}_l		576
Type $\tilde{C}'_l = {}^2\tilde{D}_{l+1}$		579
Type $\tilde{C}'_l = {}^2\tilde{A}_{2l}$	(1st description)	582
	(2nd description)	585
Type \tilde{D}_4		588
Type $\tilde{D}_l, \quad l \geq 5$		590
Type \tilde{E}_6		593
Type \tilde{E}_7		596
Type \tilde{E}_8		599
Type \tilde{F}_4		602
Type $\tilde{F}'_4 = {}^2\tilde{E}_6$		604
Type \tilde{G}_2		606
Type $\tilde{G}'_2 = {}^3\tilde{D}_4$		608
<i>Notation</i>		610
<i>Bibliography of books on Lie algebras</i>		619
<i>Bibliography of articles on Kac–Moody algebras</i>		621
<i>Index</i>		629