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R. W. Carter
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Lie Algebras of Finite and Affine Type

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R. W. CARTER

Mathematics Institute
University of Warwick

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Contents

Preface *page* xiii

1 Basic concepts 1
- 1.1 Elementary properties of Lie algebras 1
- 1.2 Representations and modules 5
- 1.3 Abelian, nilpotent and soluble Lie algebras 7

2 Representations of soluble and nilpotent Lie algebras 11
- 2.1 Representations of soluble Lie algebras 11
- 2.2 Representations of nilpotent Lie algebras 14

3 Cartan subalgebras 23
- 3.1 Existence of Cartan subalgebras 23
- 3.2 Derivations and automorphisms 25
- 3.3 Ideas from algebraic geometry 27
- 3.4 Conjugacy of Cartan subalgebras 33

4 The Cartan decomposition 36
- 4.1 Some properties of root spaces 36
- 4.2 The Killing form 39
- 4.3 The Cartan decomposition of a semisimple Lie algebra 45
- 4.4 The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ 52

5 The root system and the Weyl group 56
- 5.1 Positive systems and fundamental systems of roots 56
- 5.2 The Weyl group 59
- 5.3 Generators and relations for the Weyl group 65

6 The Cartan matrix and the Dynkin diagram 69
6.1 The Cartan matrix 69
6.2 The Dynkin diagram 72
6.3 Classification of Dynkin diagrams 74
6.4 Classification of Cartan matrices 80

7 The existence and uniqueness theorems 88
7.1 Some properties of structure constants 88
7.2 The uniqueness theorem 93
7.3 Some generators and relations in a simple Lie algebra 96
7.4 The Lie algebras $L(A)$ and $\tilde{L}(A)$ 98
7.5 The existence theorem 105

8 The simple Lie algebras 121
8.1 Lie algebras of type A_l 122
8.2 Lie algebras of type D_l 124
8.3 Lie algebras of type B_l 128
8.4 Lie algebras of type C_l 132
8.5 Lie algebras of type G_2 135
8.6 Lie algebras of type F_4 138
8.7 Lie algebras of types E_6, E_7, E_8 140
8.8 Properties of long and short roots 145

9 Some universal constructions 152
9.1 The universal enveloping algebra 152
9.2 The Poincaré–Birkhoff–Witt basis theorem 155
9.3 Free Lie algebras 160
9.4 Lie algebras defined by generators and relations 163
9.5 Graph automorphisms of simple Lie algebras 165

10 Irreducible modules for semisimple Lie algebras 176
10.1 Verma modules 176
10.2 Finite dimensional irreducible modules 186
10.3 The finite dimensionality criterion 190

11 Further properties of the universal enveloping algebra 201
11.1 Relations between the enveloping algebra and the symmetric algebra 201
11.2 Invariant polynomial functions 207
11.3 The structure of the ring of polynomial invariants 216
11.4 The Killing isomorphisms 222

11.5 The centre of the enveloping algebra 226
11.6 The Casimir element 238

12 Character and dimension formulae 241
12.1 Characters of L-modules 241
12.2 Characters of Verma modules 244
12.3 Chambers and roots 246
12.4 Composition factors of Verma modules 255
12.5 Weyl's character formula 258
12.6 Complete reducibility 262

13 Fundamental modules for simple Lie algebras 267
13.1 An alternative form of Weyl's dimension formula 267
13.2 Fundamental modules for A_l 268
13.3 Exterior powers of modules 270
13.4 Fundamental modules for B_l and D_l 274
13.5 Clifford algebras and spin modules 281
13.6 Fundamental modules for C_l 292
13.7 Contraction maps 295
13.8 Fundamental modules for exceptional algebras 303

14 Generalised Cartan matrices and Kac–Moody algebras 319
14.1 Realisations of a square matrix 319
14.2 The Lie algebra $\tilde{L}(A)$ associated with a complex matrix 322
14.3 The Kac–Moody algebra $L(A)$ 331

15 The classification of generalised Cartan matrices 336
15.1 A trichotomy for indecomposable GCMs 336
15.2 Symmetrisable generalised Cartan matrices 344
15.3 The classification of affine generalised Cartan matrices 351

16 The invariant form, Weyl group and root system 360
16.1 The invariant bilinear form 360
16.2 The Weyl group of a Kac–Moody algebra 371
16.3 The roots of a Kac–Moody algebra 377

17 Kac–Moody algebras of affine type 386
17.1 Properties of the affine Cartan matrix 386
17.2 The roots of an affine Kac–Moody algebra 394
17.3 The Weyl group of an affine Kac–Moody algebra 404

18 Realisations of affine Kac–Moody algebras 416
18.1 Loop algebras and central extensions 416
18.2 Realisations of untwisted affine Kac–Moody algebras 421
18.3 Some graph automorphisms of affine algebras 426
18.4 Realisations of twisted affine algebras 429

19 Some representations of symmetrisable Kac–Moody algebras 452
19.1 The category $\mathcal{O}$ of $L(A)$-modules 452
19.2 The generalised Casimir operator 459
19.3 Kac' character formula 466
19.4 Generators and relations for symmetrisable algebras 474

20 Representations of affine Kac–Moody algebras 484
20.1 Macdonald's identities 484
20.2 Specialisations of Macdonald's identities 491
20.3 Irreducible modules for affine algebras 494
20.4 The fundamental modules for $L(\tilde{A}_1)$ 504
20.5 The basic representation 508

21 Borcherds Lie algebras 519
21.1 Definition and examples of Borcherds algebras 519
21.2 Representations of Borcherds algebras 524
21.3 The Monster Lie algebra 530

Appendix 540
Summary pages – explanation 540
Type A_l 543
Type B_l 545
Type C_l 547
Type D_l 549
Type E_6 551
Type E_7 553
Type E_8 555
Type F_4 557
Type G_2 559

Type $\tilde{A}_1$ 561
Type $\tilde{A}'_1 = {}^2\tilde{A}_2$ (1st description) 563
(2nd description) 565
Type $\tilde{A}_l$ 567
Type $\tilde{B}_l$ 570
Type $\tilde{B}^t_l = {}^2\tilde{A}_{2l-1}$ 573
Type $\tilde{C}_l$ 576
Type $\tilde{C}^t_l = {}^2\tilde{D}_{l+1}$ 579
Type $\tilde{C}'_l = {}^2\tilde{A}_{2l}$ (1st description) 582
(2nd description) 585
Type $\tilde{D}_4$ 588
Type $\tilde{D}_l, \quad l \geq 5$ 590
Type $\tilde{E}_6$ 593
Type $\tilde{E}_7$ 596
Type $\tilde{E}_8$ 599
Type $\tilde{F}_4$ 602
Type $\tilde{F}^t_4 = {}^2\tilde{E}_6$ 604
Type $\tilde{G}_2$ 606
Type $\tilde{G}^t_2 = {}^3\tilde{D}_4$ 608

Notation 610
Bibliography of books on Lie algebras 619
Bibliography of articles on Kac–Moody algebras 621
Index 629

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Preface

Lie algebras were originally introduced by S. Lie as algebraic structures used for the study of Lie groups. The tangent space of a Lie group at the identity element has the natural structure of a Lie algebra, called by Lie the infinitesimal group. However, Lie algebras also proved to be of interest in their own right. The finite dimensional simple Lie algebras over the complex field were investigated independently by E. Cartan and W. Killing and the classification of such algebras was achieved during the decade 1890–1900. Basic ideas on the structure and representation theory of these Lie algebras were also contributed at a later stage by H. Weyl. Since then the theory of finite dimensional simple Lie algebras has found many and varied applications both in mathematics and in mathematical physics, to the extent that it is now generally regarded as one of the classical branches of mathematics.

In 1967 V. G. Kac and R. V. Moody independently introduced the Lie algebras now known as Kac–Moody algebras. The finite dimensional simple Lie algebras are examples of Kac–Moody algebras; but the theory of Kac–Moody algebras is much broader, including many infinite dimensional examples. The Kac–Moody theory has developed rapidly since its introduction and has also turned out to have applications in many areas of mathematics, including among others group theory, combinatorics, modular forms, differential equations and invariant theory. It has also proved important in mathematical physics, where it has applications to statistical physics, conformal field theory and string theory. The representation theory of affine Kac–Moody algebras has been particularly useful in such applications.

In view of these applications it seems clear that the theory of Lie algebras, of both finite and affine types, will continue to occupy a central position in mathematics into the twenty-first century. This expectation provides the motivation for the present volume, which aims to give a mathematically rigorous development of those parts of the theory of Lie algebras most relevant to the understanding of the finite dimensional simple Lie algebras and the Kac–Moody algebras of affine type. A number of books on Lie algebras are confined to the finite dimensional theory, but this seemed too restrictive for the present volume in view of the many current applications of the Kac–Moody theory. On the other hand the Kac–Moody theory needs a prior knowledge of the finite dimensional theory, both to motivate it and to supply many technical details. For this reason I have included an account both of the Cartan–Killing–Weyl theory of finite dimensional simple Lie algebras and of the Kac–Moody theory, concentrating particularly on the Kac–Moody algebras of affine type. We work with Lie algebras over the complex field, although any algebraically closed field of characteristic zero would do equally well.

I was introduced to the theory of Lie algebras by an inspiring course of lectures given by Philip Hall at Cambridge University in the late 1950s. I have given a number of lecture courses on finite dimensional Lie algebras at Warwick University, and also two lecture courses on Kac–Moody algebras. The present book has developed as a considerably expanded version of the lecture notes of these courses. The main prerequisite for study of the book is a sound knowledge of linear algebra. I have in fact aimed to make this the sole prerequisite, and to explain from first principles any other techniques which are used in the development.

The most influential book on Kac–Moody algebras is the volume *Infinite-Dimensional Lie Algebras*, third edition (1990), by V. Kac. That formidable treatise contains a development of the Kac–Moody theory presupposing a knowledge of the finite dimensional theory, and includes information on several of the applications. The present volume will not rival Kac' account for experts on Kac–Moody algebras. About half of the theory covered in the 3rd edition of Kac' book has been included. However, for those new to the Kac–Moody theory, our account may be useful in providing a gentler introduction, making use of ideas from the finite dimensional theory developed earlier in the book.

The content of the book can be summarised as follows. The basic definitions of Lie algebras, their subalgebras and ideals, representations and modules, are given in Chapter 1. In Chapter 2 the standard results are proved on the representation theory of soluble and nilpotent Lie algebras. The results on representations of nilpotent Lie algebras are used extensively in the subsequent development. The key idea of a Cartan subalgebra is introduced in Chapter 3, where the existence and conjugacy of Cartan subalgebras are proved. We make use of some ideas from algebraic geometry to prove the conjugacy of Cartan subalgebras. In Chapter 4 the Killing form is introduced and used to describe the Cartan decomposition of a semisimple Lie algebra into root spaces with respect to a Cartan subalgebra. The well-known example of the special linear Lie algebra is used to illustrate the general ideas. In Chapter 5 the Weyl group is introduced and shown to be a Coxeter group. This leads on to the definition of the Cartan matrix and the Dynkin diagram. The possible Dynkin diagrams and Cartan matrices are classified in Chapter 6, and in Chapter 7 the existence and uniqueness of a semisimple Lie algebra with a given Cartan matrix are proved. In Chapter 8 the finite dimensional simple Lie algebras are discussed individually and their root systems determined.

Chapters 9 to 13 are concerned with the representation theory of finite dimensional semisimple Lie algebras. We begin in Chapter 9 with the introduction of the universal enveloping algebra, of free Lie algebras and of Lie algebras defined by generators and relations. The finite dimensional irreducible modules for semisimple Lie algebras are obtained in Chapter 10 as quotients of infinite dimensional Verma modules with dominant integral highest weight. In Chapter 11 the enveloping algebra is studied in more detail. Its centre is shown to be isomorphic to the algebra of polynomial functions on a Cartan subalgebra invariant under the Weyl group, and to the algebra of polynomial functions on the Lie algebra invariant under the adjoint group. This algebra is shown to be isomorphic to a polynomial algebra. The properties of the Casimir element of the centre of the enveloping algebra are also discussed. These are important in subsequent applications to representation theory. Characters of modules are introduced in Chapter 12, and Weyl's character formula for the irreducible modules is proved. The fundamental irreducible modules for the finite dimensional simple Lie algebras are discussed individually in Chapter 13. Their discussion involves exterior powers of modules, Clifford algebras and spin modules, and contraction maps.

This concludes the development of the structure and representation theory of the finite dimensional Lie algebras. This development has concentrated particularly on the properties necessary to obtain the classification of the simple Lie algebras and their finite dimensional irreducible modules. Among the significant results omitted from our account are Ado's theorem on the existence of a faithful finite dimensional module, the radical splitting theorem of Levi, the theorem of Malcev and Harish-Chandra on the conjugacy of complements to the radical, and the cohomology theory of Lie algebras.

The theory of Kac–Moody algebras is introduced in Chapter 14, where the Kac–Moody algebra associated to a generalised Cartan matrix is defined. In fact there are two slightly different definitions of a Kac–Moody algebra which have been used. There is a definition in terms of generators and relations which appears the more natural, but there is a different definition, given by Kac in his book, which is more convenient when one wishes to show that a given Lie algebra is a Kac–Moody algebra. I have used the latter definition, but have included a proof that, at least for symmetrisable generalised Cartan matrices, the two definitions are equivalent.

The trichotomy of indecomposable generalised Cartan matrices into those of finite, affine and indefinite types is obtained in Chapter 15. The Kac–Moody algebras of finite type turn out to be precisely the non-trivial finite dimensional simple Lie algebras, and a classification of those of affine type is given. The important special case of symmetrisable Kac–Moody algebras is also introduced. This class includes all those of finite and affine types, and some of those of indefinite type. In Chapter 16 it is shown that symmetrisable algebras have an invariant bilinear form, which plays a key role in the subsequent development. The Weyl group and root system of a Kac–Moody algebra are also discussed. The roots divide into real roots and imaginary roots, and a remarkable theorem of Kac is proved which characterises the set of positive imaginary roots. Kac–Moody algebras of affine type are singled out for more detailed discussion in Chapter 17. In Chapter 18 it is shown how some of them can be realised in terms of a central extension of a loop algebra of a finite dimensional simple Lie algebra, whereas the remainder can be obtained as fixed point subalgebras of these under a twisted graph automorphism.

Chapters 19 and 20 are devoted to the representation theory of Kac–Moody algebras. The representations considered are those from the category $\mathcal{O}$ introduced by Bernstein, Gelfand and Gelfand. In Chapter 19 the irreducible modules in this category are classified, and their characters are obtained in Kac' character formula, a generalisation to the Kac–Moody situation of Weyl's character formula. In Chapter 20 the representations of affine Kac–Moody algebras are discussed. The remarkable identities of I. G. Macdonald are obtained by specialising the denominator of Kac' character formula, interpreted in two different ways; one as an infinite sum and the other as an infinite product. The phenomenon of strings of weights with non-decreasing multiplicities is investigated inside an irreducible module for an affine algebra.

Many of the applications of the representation theory of affine Kac–Moody algebras use the theory of vertex operators. This theory lies beyond the scope of the present volume. However, we have introduced the idea of a vertex operator in Chapter 20 with the aim of encouraging the reader to explore the subject further.

A theory of generalised Kac–Moody algebras was introduced in 1988 by R. Borcherds. These Lie algebras were introduced as part of Borcherds' proof of the Conway–Norton conjectures on the representation theory of the Monster simple group. They are now frequently called Borcherds algebras. In Chapter 21 we have given an account of Borcherds algebras, including the definition and statements of the main results concerning their structure and representation theory, but detailed proofs are not given. Many of the results on Borcherds algebras are quite similar to those for Kac–Moody algebras, but there are examples of Borcherds algebras which are quite different from Kac–Moody algebras. The best known such example is the Monster Lie algebra, which we describe in the final section.

We conclude with an appendix containing one section for each of the algebras of finite and affine types, in which the most important pieces of information about the algebra concerned are collected.

I would like to express my thanks to Roger Astley of Cambridge University Press for his encouragement to complete the half finished manuscript of this book. This was eventually achieved after I had reached the status of Emeritus Professor, and therefore had more time to devote to it. I would also like to thank my colleague Bruce Westbury for the sustained interest he has shown in this work.