1
Basic concepts

1.1 Elementary properties of Lie algebras

A **Lie algebra** is a vector space L over a field k on which a multiplication

$$L \times L \to L$$
$$(x, y) \to [xy]$$

is defined satisfying the following axioms:

(i) $(x, y) \to [xy]$ is linear in x and in y;
(ii) $[xx] = 0$ for all $x \in L$;
(iii) $[[xy]z] + [[yz]x] + [[zx]y] = 0$ for all $x, y, z \in L$.

Axiom (iii) is called the **Jacobi identity**.

Proposition 1.1 $[yx] = -[xy]$ *for all* $x, y \in L$.

Proof. Since $[x+y, x+y] = 0$ we have $[xx] + [xy] + [yx] + [yy] = 0$. It follows that $[xy] + [yx] = 0$, that is $[yx] = -[xy]$. □

Proposition 1.1 asserts that multiplication in a Lie algebra is anticommutative.

Now let H, K be subspaces of a Lie algebra L. Then $[HK]$ is defined as the subspace spanned by all products $[xy]$ with $x \in H$ and $y \in K$. Each element of $[HK]$ is a sum

$$[x_1 y_1] + \cdots + [x_r y_r]$$

with $x_i \in H$, $y_i \in K$.

Proposition 1.2 $[HK] = [KH]$ *for all subspaces* H, K *of* L.

Proof. Let $x \in H$, $y \in K$. Then $[xy] = [-y, x] \in [KH]$. This shows that $[HK] \subset [KH]$. Similarly we have $[KH] \subset [HK]$ and so we have equality. □

Proposition 1.2 asserts that multiplication of subspaces in a Lie algebra is commutative.

Example 1.3 Let A be an associative algebra over k. Thus we have a map

$$A \times A \to A$$

$$(x, y) \to xy$$

satisfying the associative law

$$(xy)z = x(yz) \qquad \text{for all } x, y, z \in A.$$

Then A can be made into a Lie algebra by defining the Lie product $[xy]$ by

$$[xy] = xy - yx$$

We verify the Lie algebra axioms. Product $[xy]$ is clearly linear in x and in y. It is also clear that $[xx] = 0$. Finally we check the Jacobi identity. We have

$$\begin{aligned}[][xy]z] &= (xy - yx)z - z(xy - yx) \\ &= xyz - yxz - zxy + zyx.\end{aligned}$$

We have similar expressions for $[[yz]x]$ and $[[zx]y]$. Hence

$$\begin{aligned}[][xy]z] + [[yz]x] + [[zx]y] &= xyz - yxz - zxy + zyx + yzx - zyx - xyz \\ &\quad + xzy + zxy - xzy - yzx + yxz = 0.\end{aligned}$$ □

The Lie algebra obtained from the associative algebra A in this way will be denoted by $[A]$.

Now let L be a Lie algebra over k. A subset H of L is called a **subalgebra** of L if H is a subspace of L and $[HH] \subset H$. Thus H is itself a Lie algebra under the same operations as L.

A subset I of L is called an **ideal** of L if I is a subspace of L and $[IL] \subset I$. We observe that the latter condition is equivalent to $[LI] \subset I$. Thus there is no distinction between left ideals and right ideals in the theory of Lie algebras. Every ideal is two-sided.

Proposition 1.4 (i) *If H, K are subalgebras of L so is $H \cap K$.*
(ii) *If H, K are ideals of L so is $H \cap K$.*

(iii) *If H is an ideal of L and K a subalgebra of L then $H+K$ is a subalgebra of L.*
(iv) *If H, K are ideals of L then $H+K$ is an ideal of L.*

Proof. (i) $H\cap K$ is a subspace of L and $[H\cap K, H\cap K]\subset[HH]\cap[KK]\subset H\cap K$. Thus $H\cap K$ is a subalgebra.
(ii) This time we have $[H\cap K, L]\subset[HL]\cap[KL]\subset H\cap K$. Thus $H\cap K$ is an ideal of L.
(iii) $H+K$ is a subspace of L. Also $[H+K, H+K]\subset[HH]+[HK]+[KH]+[KK]\subset H+K$, since $[HH]\subset H$, $[HK]\subset H$, $[KK]\subset K$. Thus $H+K$ is a subalgebra.
(iv) This time we have $[H+K, L]\subset[HL]+[KL]\subset H+K$. Thus $H+K$ is an ideal of L. □

We next introduce the idea of a factor algebra. Let I be an ideal of a Lie algebra L. Then I is in particular a subspace of L and so we can form the factor space L/I whose elements are the cosets $I+x$ for $x\in L$. $I+x$ is the subset of L consisting of all elements $y+x$ for $y\in I$.

Proposition 1.5 *Let I be an ideal of L. Then the factor space L/I can be made into a Lie algebra by defining*

$$[I+x, I+y]=I+[xy] \qquad \textit{for all } x, y\in L.$$

Proof. We must first show that this definition is unambiguous, that is if $I+x=I+x'$ and $I+y=I+y'$ then $I+[xy]=I+[x'y']$.

Now $I+x=I+x'$ implies that $x=x'+i_1$ for some $i_1\in I$. Similarly $I+y=I+y'$ implies $y=y'+i_2$ for some $i_2\in I$. Thus

$$\begin{aligned} I+[xy] &= I+[x'+i_1, y'+i_2] \\ &= I+[i_1y']+[x'i_2]+[i_1i_2]+[x'y'] \\ &= I+[x'y'] \end{aligned}$$

since $[i_1y']$, $[x'i_2]$, $[i_1i_2]$ all lie in I. Thus our multiplication is well defined. We also have

$$[I+x, I+x]=I+[xx]=I$$

and the Jacobi identity in L/I clearly follows from the Jacobi identity in L. □

Now suppose we have two Lie algebras L_1, L_2 over k. A map $\theta : L_1 \to L_2$ is called a **homomorphism of Lie algebras** if θ is linear and

$$\theta[xy] = [\theta x, \theta y] \qquad \text{for all } x, y \in L_1.$$

The map $\theta : L_1 \to L_2$ is called an **isomorphism of Lie algebras** if θ is a bijective homomorphism of Lie algebras. The Lie algebras L_1, L_2 are said to be **isomorphic** if there exists an isomorphism $\theta : L_1 \to L_2$.

Proposition 1.6 *Let $\theta : L_1 \to L_2$ be a homomorphism of Lie algebras. Then the image of θ is a subalgebra of L_2, the kernel of θ is an ideal of L_1 and $L_1/\ker\theta$ is isomorphic to* $\operatorname{im}\theta$.

Proof. $\operatorname{im}\theta$ is a subspace of L_2. Moreover for x, y in L_1 we have

$$[\theta(x), \theta(y)] = \theta[xy] \in \operatorname{im}\theta.$$

Hence $\operatorname{im}\theta$ is a subalgebra of L_2.

Now $\ker\theta$ is a subspace of L_1. Let $x \in \ker\theta$ and $y \in L_1$. Then

$$\theta[xy] = [\theta(x), \theta(y)] = [0, \theta(y)] = 0.$$

Hence $[xy] \in \ker\theta$ and so $\ker\theta$ is an ideal of L_1.

Now let $x, y \in L_1$. We consider when $\theta(x)$ is equal to $\theta(y)$. We have

$$\begin{aligned}\theta(x) = \theta(y) \ &\Leftrightarrow \theta(x - y) = 0 \Leftrightarrow x - y \in \ker\theta \\ &\Leftrightarrow \ker\theta + x = \ker\theta + y.\end{aligned}$$

This shows that there is a bijective map $\theta(x) \to \ker\theta + x$ between $\operatorname{im}\theta$ and $L_1/\ker\theta$. We show this bijection is an isomorphism of Lie algebras. It is clearly linear. Moreover given $x, y, z \in L_1$ we have

$$\begin{aligned}[\theta(x), \theta(y)] = \theta(z) &\Leftrightarrow \theta[xy] = \theta(z) \\ &\Leftrightarrow \ker\theta + [xy] = \ker\theta + z \\ &\Leftrightarrow [\ker\theta + x, \ker\theta + y] = \ker\theta + z.\end{aligned}$$

Thus the bijection preserves Lie multiplication, so is an isomorphism of Lie algebras. □

Proposition 1.7 *Let I be an ideal of L and H a subalgebra of L. Then*

(i) *I is an ideal of $I + H$.*
(ii) *$I \cap H$ is an ideal of H.*
(iii) *$(I + H)/I$ is isomorphic to $H/(I \cap H)$.*

Proof. We recall from Proposition 1.4 that $I \cap H$ and $I+H$ are subalgebras. We have $[I, I+H] \subset [IL] \subset I$, thus I is an ideal of $I+H$. Also $[I \cap H, H] \subset [IH] \cap [HH] \subset I \cap H$, thus $I \cap H$ is an ideal of H.

Let $\theta : H \to (I+H)/I$ be defined by $\theta(x) = I + x$. This is clearly a linear map, and is also evidently a homomorphism of Lie algebras. It is surjective since each element of $(I+H)/I$ has form $I+x$ for some $x \in H$. Finally its kernel is the set of $x \in H$ for which $I+x = I$, that is $I \cap H$. Thus $(I+H)/I$ is isomorphic to $H/(I \cap H)$ by Proposition 1.6. □

1.2 Representations and modules

Let $M_n(k)$ be the associative algebra of all $n \times n$ matrices over the field k and let $[M_n(k)]$ be the corresponding Lie algebra. This is often called the **general linear Lie algebra** of degree n over k and we write

$$\mathfrak{gl}_n(k) = [M_n(k)].$$

We have $\dim \mathfrak{gl}_n(k) = n^2$.

A **representation** of a Lie algebra L over k is a homomorphism of Lie algebras

$$\rho : L \to \mathfrak{gl}_n(k)$$

for some n, and ρ is called a representation of degree n. Two representations ρ, ρ' of degree n are called **equivalent** if there exists a non-singular $n \times n$ matrix T such that

$$\rho'(x) = T^{-1} \rho(x) T \qquad \textit{forall } x \in L.$$

A **left L-module** is a vector space V over k together with a multiplication

$$L \times V \to V$$

$$(x, v) \to xv$$

satisfying the axioms:

(i) $(x, v) \to xv$ is linear in x and in v;
(ii) $[xy]v = x(yv) - y(xv)$ for all $x, y \in L$ and $v \in V$.

Suppose V is a finite dimensional L-module. Let $e_1, \dots, e_n$ be a basis of V. Let

$$xe_j = \sum_i \rho_{ij}(x) e_i$$

with $\rho_{ij}(x) \in k$ and let $\rho(x) = (\rho_{ij}(x))$. Then ρ is a representation of L. For we have

$$\begin{aligned}
[xy]e_j &= x(ye_j) - y(xe_j) \\
&= x\left(\sum_k \rho_{kj}(y)e_k\right) - y\left(\sum_k \rho_{kj}(x)e_k\right) \\
&= \sum_k \rho_{kj}(y)xe_k - \sum_k \rho_{kj}(x)ye_k \\
&= \sum_k \rho_{kj}(y)\left(\sum_i \rho_{ik}(x)e_i\right) - \sum_k \rho_{kj}(x)\left(\sum_i \rho_{ik}(y)e_i\right) \\
&= \sum_i \left(\sum_k (\rho_{ik}(x)\rho_{kj}(y) - \rho_{ik}(y)\rho_{kj}(x))\right) e_i \\
&= \sum_i (\rho(x)\rho(y) - \rho(y)\rho(x))_{ij}\, e_i.
\end{aligned}$$

Thus $\rho[xy] = \rho(x)\rho(y) - \rho(y)\rho(x) = [\rho(x), \rho(y)]$ and ρ is a representation of L.

Suppose now we take a second basis $f_1, \ldots, f_n$ of V. Let ρ' be the representation of L obtained from this basis. Then ρ' is equivalent to ρ. For there exists a non-singular $n \times n$ matrix T such that

$$f_j = \sum_i T_{ij}e_i.$$

Thus we have

$$xf_j = \sum_k T_{kj}xe_k = \sum_k T_{kj}\left(\sum_i \rho_{ik}(x)e_i\right) = \sum_i \left(\sum_k \rho_{ik}(x)T_{kj}\right) e_i.$$

On the other hand

$$xf_j = \sum_k \rho'_{kj}(x)f_k = \sum_k \rho'_{kj}(x)\left(\sum_i T_{ik}e_i\right) = \sum_i \left(\sum_k T_{ik}\rho'_{kj}(x)\right) e_i.$$

It follows that $\rho(x)T = T\rho'(x)$, that is $\rho'(x) = T^{-1}\rho(x)T$ for all $x \in L$. Hence the representation ρ' is equivalent to ρ. □

Example 1.8 L is itself a left L-module.

The left action of L on L is defined as $x \cdot y = [xy]$. Then we have

$$[[xy]z] = [x[yz]] - [y[xz]]$$

which is a consequence of the Jacobi identity. This shows that L is a left L-module. This is called the **adjoint module**. We define $\mathrm{ad}\, x : L \to L$ by

$$\mathrm{ad}\ x \cdot y = [xy] \qquad \text{for } x, y \in L.$$

Then we have

$$\mathrm{ad}[xy] = \mathrm{ad}\ x\ \mathrm{ad}\ y - \mathrm{ad}\ y\ \mathrm{ad}\ x. \qquad \square$$

Now let V be a left L-module, U be a subspace of V and H a subspace of L. We define HU to be the subspace of V spanned by all elements of the form xu for $x \in H$, $u \in U$.

A **submodule** of V is a subspace U of V such that $LU \subset U$. In particular V is a submodule of V and the zero subspace $O = \{0\}$ is a submodule of V. A **proper submodule** of V is a submodule distinct from V and O.

An L-module V is called **irreducible** if it has no proper submodules. V is called **completely reducible** if it is a direct sum of irreducible submodules. V is called **indecomposable** if V cannot be written as a direct sum of two proper submodules. Of course every irreducible L-module is indecomposable, but the converse need not be true.

We may also define right L-modules, but we shall mainly work with left L-modules, and L-modules will be assumed to be left L-modules unless otherwise stated.

1.3 Abelian, nilpotent and soluble Lie algebras

A Lie algebra L is **abelian** if $[LL] = O$. Thus $[xy] = 0$ for all $x, y \in L$ when L is abelian.

Given any Lie algebra L we define the powers of L by

$$L^1 = L, \qquad L^{n+1} = [L^n L] \qquad \text{for } n \geq 1.$$

Thus L is abelian if and only if $L^2 = O$.

Proposition 1.9 *L^n is an ideal of L. Also*

$$L = L^1 \supset L^2 \supset L^3 \supset \cdots.$$

Proof. We first observe that if I, J are ideals of L then $[IJ]$ is also an ideal of L. For let $x \in I$, $y \in J$, $z \in L$. Then

$$[[xy]z] = [x[yz]] - [y[xz]] \in [IJ].$$

It follows that L^n is an ideal of L for each $n > 0$. Thus we have

$$L^{n+1} = [L^n L] \subset L^n. \qquad \square$$

A Lie algebra L is called **nilpotent** if $L^n = O$ for some $n \geq 1$. Thus every abelian Lie algebra is nilpotent. It is clear that every subalgebra and every factor algebra of a nilpotent Lie algebra are nilpotent.

We now consider a different kind of powers of L. We define

$$L^{(0)} = L, \quad L^{(n+1)} = [L^{(n)}, L^{(n)}] \qquad \text{for } n \geq 0.$$

Proposition 1.10 *$L^{(n)}$ is an ideal of L. Also*

$$L = L^{(0)} \supset L^{(1)} \supset L^{(2)} \supset \cdots.$$

Proof. $L^{(n)}$ is an ideal of L since the product of two ideals is an ideal. Also

$$L^{(n+1)} = [L^{(n)}, L^{(n)}] \subset L^{(n)}. \qquad \square$$

A Lie algebra L is called **soluble** if $L^{(n)} = O$ for some $n \geq 0$.

Proposition 1.11 *(a) $[L^m L^n] \subset L^{m+n}$ for all $m, n \geq 1$. (b) $L^{(n)} \subset L^{2^n}$ for all $n \geq 0$. (c) Every nilpotent Lie algebra is soluble.*

Proof. (a). We use induction on n. The result is clear if $n = 1$. Suppose it is true for $n = r$. Then

$$\begin{aligned} [L^m L^{r+1}] &= [L^m [L^r L]] = [[L^r L] L^m] \\ &\subset [[L L^m] L^r] + [[L^m L^r] L] \qquad \text{by the Jacobi identity} \\ &\subset [L^{m+1} L^r] + [[L^m L^r] L] \\ &\subset L^{m+r+1} \qquad \text{by inductive hypothesis.} \end{aligned}$$

Thus the result holds for $n = r + 1$, so for all n.

(b). We again use induction on n. The result is clear if $n = 1$. Suppose it is true for $n = r$. Then

$$L^{(r+1)} = [L^{(r)} L^{(r)}] \subset [L^{2^r} L^{2^r}] \subset L^{2^{r+1}}$$

by (a). Thus the result holds for $n = r + 1$, so for all n.

(c). Suppose L is nilpotent. Then $L^{2^n} = O$ for n sufficiently large. Hence $L^{(n)} = O$ by (b) and so L is soluble. $\square$

It is clear that every subalgebra and every factor algebra of a soluble Lie algebra are soluble.

Proposition 1.12 *Suppose I is an ideal of L and both I and L/I are soluble. Then L is soluble.*

Proof. Since L/I is soluble we have $(L/I)^{(n)} = O$ for some n. This implies $L^{(n)} \subset I$. Since I is soluble we have $I^{(m)} = O$ for some m. Hence

$$L^{(n+m)} = (L^{(n)})^{(m)} \subset I^{(m)} = O$$

and so L is soluble. □

Proposition 1.13 *Every finite dimensional Lie algebra L contains a unique maximal soluble ideal R. Also L/R contains no non-zero soluble ideal.*

Proof. Let I, J be soluble ideals of L. Then $I + J$ is also an ideal of L and $(I + J)/I$ is isomorphic to $J/(I \cap J)$ by Proposition 1.7. Now J is soluble, thus $J/(I \cap J)$ is soluble and so $(I + J)/I$ is soluble. Since I is soluble we see that $I + J$ is soluble by Proposition 1.12. Thus the sum of two soluble ideals of L is a soluble ideal. It follows that L has a unique maximal soluble ideal R.

If I/R is a soluble ideal of L/R then I is a soluble ideal of L by Proposition 1.12. Hence $I = R$ and $I/R = O$. □

The ideal R is called the **soluble radical** of L. A Lie algebra L is called **semisimple** if $R = O$. Thus L is semisimple if and only if L has no non-zero soluble ideal.

L is called **simple** if L has no proper ideal, that is no ideal other than L and O.

Suppose L is a Lie algebra of dimension 1 over k. Then L has a basis $\{x\}$ with 1 element. Since $[xx] = 0$ we have $L^2 = O$. Thus L is abelian. We see that any two 1-dimensional Lie algebras over k are isomorphic. Of course any such Lie algebra is simple, because L has no proper subspaces. The 1-dimensional Lie algebra is called the **trivial simple Lie algebra**. A non-trivial simple Lie algebra is a simple Lie algebra L with $\dim L > 1$.

Proposition 1.14 *Each non-trivial simple Lie algebra is semisimple.*

Proof. Suppose L is simple but not semisimple. Then the soluble radical R satisfies $R \neq O$. Since R is an ideal of L this implies $R = L$. Thus L is soluble.

Hence $L^{(n)} = O$ for some $n \geq 0$. This implies that $L^{(1)} \neq L$ since $L^{(1)} = L$ would imply $L^{(n)} = L$ for all n. Now $L^{(1)}$ is an ideal of L, hence $L^{(1)} = O$ since L is simple. Thus $[LL] = O$. But then every subspace of L is an ideal of L. Since L is simple L has no proper subspaces, so $\dim L = 1$. Thus the only simple Lie algebra which is not semisimple is the trivial simple Lie algebra. □