Introduction

TO SET THE STAGE let $\mathbb{F}$ be a field (in much of what follows $\mathbb{F}$ will be finite, e.g., $\mathbb{F}_2$, the field with 2 elements) and $n \in \mathbb{N}$ a positive integer. Denote by $V = \mathbb{F}^n$ the n-dimensional vector space over $\mathbb{F}$, and by $\mathbb{F}[V]$ the graded algebra of homogeneous polynomial functions on V. To be specific, $\mathbb{F}[V]$ is the symmetric algebra $S(V^*)$ on the vector space V^* dual to V. Since graded commutation rules play no role here we will grade this as an algebraist would, i.e., putting the linear forms in degree 1 no matter what the characteristic of the ground field $\mathbb{F}$. The homogeneous component of $\mathbb{F}[V]$ of degree d will be denoted by $\mathbb{F}[V]_d$. If we need a notation for a basis of V^* we will use $z_1, \ldots, z_n$; the corresponding basis for V will be denoted by $u_1, \ldots, u_n$. For up to three variables we will most often write x, y, z, respectively u, v, w for the variables and their duals. Recall that a graded vector space, algebra, or module is said to have **finite type** if the homogeneous components are all finite dimensional vector spaces.

DEFINITION: *Let H be a commutative graded connected algebra of finite type over the field $\mathbb{F}$. We say that H is a* **Poincaré duality algebra** *of* **formal dimension** d *if*

(i) $H_i = 0$ *for* $i > d$,

(ii) $\dim_{\mathbb{F}}(H_d) = 1$,

(iii) the pairing $H_i \otimes H_{d-i} \longrightarrow H_d$ *given by multiplication is nonsingular, i.e., an element* $a \in H_i$ *is zero if and only if* $a \cdot b = 0 \in H_d$ *for all* $b \in H_{d-i}$.

If H is a Poincaré duality algebra we write f-dim(H) for the formal dimension of H. If the formal dimension is d and $[H]$ in H_d is nonzero, then $[H]$ is referred to as a **fundamental class** for H. Fundamental classes are determined only up to multiplication by a nonzero element of $\mathbb{F}$.

The notion of Poincaré duality comes from the study of closed manifolds in algebraic topology, and goes back at least to H. Poincaré; see e.g. [77] Section 69. Apart from the graded commutation rules the cohomology of a closed oriented manifold with field coefficients is the prototypical example of a Poincaré duality algebra. However, Poincaré duality algebras also appear quite naturally in invariant theory as rings of coinvariants of groups whose rings of invariants are polynomial algebras. Indeed, the less than complete understanding of the role of Poincaré duality algebras in invariant theory is part of the motivation for this study. We explain this next.

In characteristic zero, or more generally in the nonmodular case, i.e., where the order $|G|$ of G is invertible in the ground field, it is well known (see e.g. [80] or [87] Section 7.4) that pseudoreflection groups (better said, pseudoreflection representations) are characterized by the fact that their invariant rings are polynomial algebras. This is known to fail in the modular case: the ring of invariants of a reflection group need not be a polynomial algebra (see e.g. [100] or [87] Section 7.4 Example 4).

If $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a representation of a finite group G over the field $\mathbb{F}$ for which the ring of invariants $\mathbb{F}[V]^G$ is a polynomial algebra then the ring of coinvariants $\mathbb{F}[V]_G$ is a Poincaré duality quotient of $\mathbb{F}[V]$ ([87] Theorem 6.5.1). Such a ring of coinvariants is therefore a very special type of Poincaré duality algebra, viz., a complete intersection. A theorem of R. Steinberg [98] (as formulated by R. Kane [40], [41] Chapter VII) says: if $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a representation of a finite group over a field $\mathbb{F}$ *of characteristic zero*, then the ring of coinvariants $\mathbb{F}[V]_G$ is a Poincaré duality algebra if and only if G is a pseudoreflection group. Although Steinberg's proof, as well as Kane's, makes central use of the fact that the characteristic of $\mathbb{F}$ is zero and not just relatively prime to the order of the group, as would seem more natural, T.-C. Lin [47] has recently removed the need for this extra assumption and shown the result holds in the nonmodular case.

It is not known what the best extension of Steinberg's theorem to fields of nonzero characteristic might look like. Several variations of its hypotheses and conclusion are possible. There is an ad hoc, characteristic free, proof of the result as originally stated if one restricts to the case of two variables [92]. Examples in the modular case show that in more variables the original statement can be false. For example, consider the tautological representation of the alternating group A_n over a field of characteristic p less than or equal to n. Then $\mathbb{F}[z_1, \ldots, z_n]_{A_n}$ is a Poincaré duality algebra, but the ring of invariants is not a polynomial algebra (see [25], Section 11, [83] Section 5, and [93] Section 2). It is also not clear that in the modular case a ring of coinvariants which is a Poincaré duality algebra must be a complete intersection, though this is again correct for $n = 2$ by [102]. Other places where Poincaré duality algebras appear in connection with invariant theory may be found in [19] and [41].

An important feature of an algebra over a field of characteristic $p \neq 0$ is the operation of raising an element to the p-th power. This is loosely referred to as the Frobenius map. The Steenrod operations and the Steenrod algebra represent one way to organize information hidden in the Frobenius homomorphism provided the ground field is finite. Let $\mathbb{F}_q$ be the Galois field with $q = p^\nu$ elements, $V = \mathbb{F}_q^n$, and define[1]

$$\mathcal{P}(\xi) : \mathbb{F}_q[V] \longrightarrow \mathbb{F}_q[V][[\xi]]$$

by the rules

(i) $\mathcal{P}(\xi)$ is $\mathbb{F}$-linear,
(ii) $\mathcal{P}(\xi)(v) = v + v^q\xi$ for $v \in V^*$,
(iii) $\mathcal{P}(\xi)(u \cdot w) = \mathcal{P}(\xi)(u) \cdot \mathcal{P}(\xi)(w)$ for $u,\ w \in \mathbb{F}_q[V]$,
(iv) $\mathcal{P}(\xi)(1) = 1$.

$\mathcal{P}(\xi)$ is a ring homomorphism of degree 0 if we agree ξ has degree $(1-q)$. By separating out homogeneous components we obtain $\mathbb{F}_q$-linear maps

$$\mathcal{P}^i : \mathbb{F}_q[V] \longrightarrow \mathbb{F}_q[V]$$

by the requirement

$$\mathcal{P}(\xi)(f) = \sum_{i=0}^{\infty} \mathcal{P}^i(f)\xi^i.$$

The maps $\mathcal{P}^i$ may be assembled into an algebra called the Steenrod algebra $\mathscr{P}^*$ of the Galois field $\mathbb{F}_q$ (see e.g. [87] Chapters 10 and 11). The strength of these operations lies in the fact that they commute with the action of $\mathrm{GL}(V)$ on $\mathbb{F}_q[V]$ and satisfy the unstability conditions

$$\mathcal{P}^i(f) = \begin{cases} f^q & i = \deg(f) \\ 0 & i > \deg(f) \end{cases}$$

for all $f \in \mathbb{F}_q[V]$. The first unstability condition is a *nontriviality* condition, the second a *triviality* condition, and together they impose a very rigid restriction on $\mathbb{F}_q[V]^G$ as well as on which Poincaré duality quotients of $\mathbb{F}_q[V]$ can arise as $\mathbb{F}_q[V]_G$ for some representation $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F}_q)$. For example, using ideas from algebraic topology one can define invariants of such quotients called Wu classes. In Appendix B of [60] it is shown that the Wu classes (see Section IV.2) of a ring of coinvariants which is a Poincaré duality algebra are always trivial. Not all Poincaré duality quotients with an unstable action of the Steenrod operations have this property.

[1] If A is a ring then $A[[\xi]]$ denotes the ring of formal power series over A in the variable ξ.

Steinberg's paper [98] contains a number of other results on rings of coinvariants that are Poincaré duality algebras, such as how to construct a fundamental class. Again, as they stand, the proofs work only in characteristic zero. Some of these results have been extended to nonzero characteristic [91], but by no means all. This unsatisfactory state of affairs suggested that a systematic study of Poincaré duality quotients of $\mathbb{F}_q[V]$ admitting Steenrod operations satisfying the unstability conditions might be fruitful. We begin such a study here.

The first step in our program led us to Macaulay's theory of irreducible ideals in polynomial algebras. Due to the enormous changes in terminology since [49] was written, this theory is not as accessible as it might be. We present a treatment in current parlance and extend it to encompass the extra structure of a Steenrod algebra action when the ground field is finite. In doing so we uncover a very unexpected connection between $\mathscr{P}^*$-Poincaré duality quotients of $\mathbb{F}_q[V]$ and the so called *Hit Problems*,[2] e.g., the determination of the $\mathscr{P}^*$-indecomposable elements in $\mathbb{F}_q[V]$. This relationship is particularly appealing if formulated along the lines of M. C. Crabb and J. R. Hubbuck [16]. Applications to this problem based on what we have learned about $\mathscr{P}^*$-Poincaré duality algebras appear at various places in Parts III and VI, and in detail in Part V, where we work out a number of special cases.

There are many other motivations for studying Poincaré duality quotients of $\mathbb{F}[V]$. Let us just mention one more: it has its origins in algebraic topology and connects up with certain problems in invariant theory which we have not discussed here (see [44]). We paraphrase from the introduction to [1]. *"Recently*, in studying the coinvariants of reflection groups, *I had occasion to consider the formulae of Thom and Wu* [111] . . . *although these formulae are simple and attractive, I did not feel that they gave me that complete understanding that I sought."* In [1] J. F. Adams proves these formulae by constructing a universal example that is no longer a Poincaré duality algebra[3] but is an inverse limit of rings of coinvariants (see e.g. [10] or [44]) and verifying certain other formulae in this new object. We would like to understand why the formulae of Thom and Wu are *also* consequences of

[2] We use the expression *Hit Problem(s)* as in [108] Section 7: *quite generally, if M is a graded module over the positively graded algebra A over the field $\mathbb{F}$, we say that $u \in M$ is* **hit** *if there are elements $u_1, \ldots, u_k \in M$ and $a_1, \ldots, a_k \in A$ with $\deg(a_1), \ldots, \deg(a_k) > 0$ with $u = a_1u_1 + \cdots + a_ku_k$. The elements of M that are hit form the A-submodule $\overline{A} \cdot M$, and the quotient $M/\overline{A} \cdot M$ the module of A-indecomposable elements $\mathbb{F} \otimes_A M$. Hit Problem(s)* refer to the characterization of elements of M that are hit or not, e.g., in the case of the Steenrod algebra $\mathscr{P}^*$ acting on $\mathbb{F}_q[z_1, \ldots, z_n]$ finding conditions on a monomial that assure it is $\mathscr{P}^*$-indecomposable.

[3] In fact it isn't even Noetherian.

their validity in the Poincaré duality quotients of $\mathbb{F}_q[V]$ admitting an unstable Steenrod algebra action. The action of the Steenrod algebra in these cases is completely formal, it being a consequence of unstability and the Cartan formula. Why should these determine the formulae for arbitrary $\mathscr{P}^*$-unstable Poincaré duality algebras?

THE MANUSCRIPT divides naturally into several parts. The material in Part I is largely expository. There we explain the connection between Poincaré duality algebras, Gorenstein algebras, and irreducible ideals. Several different characterizations of irreducible ideals are given and a variety of methods to construct them are presented. We specialize and refine these results and constructions to the main case of interest for us, namely the Poincaré duality quotient algebras of $\mathbb{F}[V]$. A method is developed for counting the number of isomorphism classes of such quotients over a Galois field. It is based on invariant theory and is applied to count the number of such quotients of $\mathbb{F}_2[x, y]$.

Part II reformulates a number of results from [49] in modern language. In Section II.2 we present Macaulay's concept of inverse systems (which are called dual systems here) in the language of Hopf algebras and derive a number of results that will be useful in later sections. This is then illustrated with examples and connected with the classical form problem of ninteenth century invariant theory. A fundamental tool for making computations, the $K \subset L$ paradigm, appears in Section II.5. Section II.6 contains a result first proved in the ungraded case by R. Y. Sharp. Namely, a Frobenius power of an irreducible ideal in a regular local ring is again irreducible. Careful study of his proof has allowed us to give a proof adapted to the graded case. We use this to construct new Poincaré duality quotients from existing ones. As a bonus, the proof in the graded case yields formulae for a fundamental class and a generator for the dual principal system of the new Poincaré duality quotient.

In Part III we restrict the ground field to a Galois field. Here the Frobenius homomorphism provides us with an additional structure which we organize[4] into the Steenrod algebra $\mathscr{P}^*$ of $\mathbb{F}_q$ whose elements are called Steenrod operations. Section III.1 introduces the Steenrod algebra $\mathscr{P}^*$ and in Section III.2 we rework Macaulay's Double Annihilator Theorem

[4] This is by no means the only way to extract information from the Frobenius homomorphism: see [73] and [71] for a different approach altogether.

(see Section II.2) in this enhanced context. Wu classes are invariants of the action of the Steenrod algebra on a Poincaré duality algebra; they are introduced in Section III.3. In the case of a Poincaré duality quotient of $\mathbb{F}_q[V]$ their vanishing is related by means of Macaulay's $\mathscr{P}^*$-Double Annihilator Theorem to the problem of computing the invariants $\Gamma(V)^{\mathscr{P}^*}$ of the Steenrod algebra acting[5] on the dual divided power algebra $\Gamma(V)$. These results culminate in a surprising connection with a *Hit Problem*, namely of finding a minimal set of generators of $\mathbb{F}_q[V]$ as a $\mathscr{P}^*$-module.

In Part IV we investigate the structure of algebras of coinvariants that are Poincaré duality quotients of $\mathbb{F}_q[z_1, \dots, z_n]$. A ring of coinvariants over a finite field is an unstable algebra over the Steenrod algebra. If a ring of invariants is a polynomial algebra then the corresponding ring of coinvariants is a Poincaré duality algebra, so potentially has nonzero Wu classes. The Dickson and symmetric coinvariants are such algebras and they are examined in detail. We determine fundamental classes, Macaulay duals, and by explicit computations show that the Wu classes are trivial. The relation of these algebras to the *Hit Problem* is explained here, with detailed computations carried out in Part V. In [60] S. A. Mitchell showed that a complete intersection algebra with unstable Steenrod algebra action that is a quotient of $\mathbb{F}_q[V]$ has trivial Wu classes. We present a variant of his proof in Section IV.2 that exploits our computations with the Dickson coinvariants and the Adams and Wilkerson/Neusel Imbedding Theorem, [66] Theorem 7.4.4.

Part V contains some detailed applications to the *Hit Problem* for $\mathbb{F}_2[V]$. We have confined ourselves to the choice of $\mathbb{F}_2$ as ground field to keep the gymnastics with binomial coefficients mod p within reasonable bounds. Section V.1 examines ideals generated by powers of the Dickson polynomials $\mathbf{d}_{2,0}, \mathbf{d}_{2,1} \in \mathbb{F}_2[x, y]$ where we determine which of these ideals are $\mathscr{A}^*$-invariant, and precisely which amongst those have a Poincaré duality quotient with trivial Wu classes. In Section V.2 we show that representatives for the fundamental classes of the latter, together with the so-called spikes (see [82]), provide a vector space basis for the $\mathscr{A}^*$-indecomposable elements of $\mathbb{F}_2[x, y]$. Some topological questions arising from these results are considered in [56]. Section V.3 extends this line of investigation to three variables. Our work with ideals generated by powers of Dickson polynomials culminates in a complete list of such ideals in an arbitrary number of variables that are $\mathscr{A}^*$-invariant and determines which of these

[5] If one introduces the **total Steenrod operator** $\mathscr{P} = 1 + \mathscr{P}^1 + \cdots + \mathscr{P}^k + \cdots$, then $\mathscr{P}$ induces an *ungraded* action of the integers $\mathbb{Z}$ on $\Gamma(V)$ and $\Gamma(V)^{\mathscr{P}^*}$ is the subalgebra of elements invariant under this action.

have trivial Wu classes. We close this part by commencing a study of the ideals generated by powers of Stiefel–Whitney classes; these results are not as complete as for the case of powers of Dickson polynomials. Section V.5 examines ideals generated by powers of the Stiefel–Whitney classes $w_2, w_3, w_4 \in \mathbb{F}_2[x, y, z]$, which provide some interesting examples of $\mathscr{A}^*$-indecomposable monomials.

Part VI grew out of a simple observation concerning the relation of the *Hit Problems* for $\mathbb{F}_q[z_1, \ldots, z_n]$ and $\mathbf{D}(n)$. We decided to develop a more general context in which to present this observation. The results in Part VI center around a discussion of the classical *lying over* relation, not for prime ideals, but for irreducible and regular ideals. We first introduce Macaulay's *inverse systems* in their original form using inverse polynomials. This we use to explain the *catalecticant matrices* which provide a tool to make computations. We illustrate this in a pair of examples in Section VI.2. We use the inverse system formulation of Macaulay's theory, together with our results on lying over for irreducible and regular ideals to prove a variety of *change of rings* results (see Sections VI.5 and VI.7). This allows us to better study the relation between two Poincaré duality quotients $\mathbb{F}[V]/I$ and $\mathbb{F}[V]/J$ under the hypothesis that $I \subset J$. We obtain a number of results relating fundamental classes and generators for the inverse principal systems. In the special case that $\mathbb{F} = \mathbb{F}_q$ and the ideals are closed under the action of the Steenrod algebra we enhance these results to include Steenrod operations, and apply them to the *Hit Problems* for the Dickson and other algebras.

For unexplained terminology or notation please see the index of notation, [68], or [87].

Part I
Poincaré duality quotients

In this part we collect a certain amount of background material. There is basically not much new here, but these results are scattered throughout the literature, and where they do appear, they rarely do so in the form we need. We emphasize at the outset that, unless explicitly mentioned to the contrary, all algebras considered here are commutative, graded, connected algebras of finite type over a field. The component of degree k of a graded object X is denoted by X_k. We start by reviewing some definitions and then develop the relation between Gorenstein algebras, Poincaré duality algebras, and irreducible ideals. The study of irreducible ideals seems to have fallen out of favor with time so it is difficult to locate adequate references for some results. We therefore include a fair number of elementary proofs.

Once the basic objects of study have been introduced and their elementary properties developed we use them to present several different ways to construct Poincaré duality algebras that are quotients of a polynomial algebra. In the final section of this part we examine the problem of counting such Poincaré duality quotients up to isomorphism in the case of a finite ground field. This leads to an interesting invariant theoretic problem.

I.1 Poincaré duality, Gorenstein algebras, and irreducible ideals

If A is a commutative graded connected algebra over a field $\mathbb{F}$ we denote by $\overline{A}$ the **augmentation ideal** of A: this is the ideal of all homogeneous elements of strictly positive degree. This notation becomes a bit cumbersome for $\mathbb{F}[x, y, z]$, viz., $\overline{\mathbb{F}[x, y, z]}$, so if the algebra A is clear from the context we introduce the alternative notation $\mathfrak{m}$ for its augmentation ideal.

Note that this is the unique *graded* maximal ideal in A. If H is a commutative graded connected algebra over a field $\mathbb{F}$ and $[H] \in H$ is an element of degree d, then it is easy to see that H is a Poincaré duality algebra of formal dimension d with fundamental class $[H]$ if and only if $H_i = 0$ for $i > d$, $\mathrm{Ann}_H([H]) = \overline{H}$, and $\mathrm{Ann}_H(\overline{H}) = \mathbb{F} \cdot [H]$.

The study of Poincaré duality algebras is permeated with *double duality* results of one form or another. Here is one of the most basic of these (see e.g. [18] page 416 or [63]).

LEMMA I.1.1: *Let H be a Poincaré duality algebra over the field $\mathbb{F}$ and $I \subset H$ an ideal. Then* $\mathrm{Ann}_H(\mathrm{Ann}_H(I)) = I$.

PROOF: Let the formal dimension of H be d and choose a fundamental class $[H] \in H_d$. Define a bilinear pairing

$$<- \mid -> : H \times H \longrightarrow \mathbb{F}$$

by

$$<a \mid b> = \begin{cases} 0 & \text{if } \deg(a) + \deg(b) \neq d \\ \lambda & \text{if } \deg(a) + \deg(b) = d \text{ and } a \cdot b = \lambda[H]. \end{cases}$$

Note that this pairing is symmetric, middle associative, and nondegenerate, i.e.,

$$\begin{aligned} <a \mid b> &= <b \mid a> \quad \forall\ a,\ b \in H \\ <a \cdot c \mid b> &= <a \mid c \cdot b> \quad \forall\ a,\ c,\ b \in H \\ <a \mid b> = 0 \quad \forall\ b \in H &\iff a = 0. \end{aligned}$$

We claim that $\mathrm{Ann}_H(I) = \{h \in H \mid <h \mid I> = 0\}$. To see this suppose that $u \in \mathrm{Ann}_H(I)$ and $w \in I$. If $\deg(u) + \deg(w) \neq d$ then $<u \mid w> = 0$ by definition. If $\deg(u) + \deg(w) = d$, then since u annihilates I, we have $u \cdot w = 0 = 0 \cdot [H]$, so $<u \mid w> = 0$. Therefore it follows that $\mathrm{Ann}_H(I) \subseteq \{h \in H \mid <h \mid I> = 0\}$. On the other hand, if $u \in \{h \in H \mid <h \mid I> = 0\}$, and $w \in I$, then for any $x \in H_{d-(\deg(u)+\deg(w))}$ we have $w \cdot x \in I$, so

$$0 = <u \mid w \cdot x> = <u \cdot w \mid x>.$$

Hence $u \cdot w$ annihilates $H_{d-(\deg(u)+\deg(w))}$ so by Poincaré duality $u \cdot w = 0$, therefore $u \in \mathrm{Ann}_H(I)$. This establishes the claim. From this the lemma follows using elementary facts about nondegenerate bilinear forms (see for example [35] Chapter V Section 3). □

The tensor product $H = H' \otimes H''$ of two Poincaré duality algebras H' and H'' is again a Poincaré duality algebra: if $[H'] \in H'_{d'}$ and $[H''] \in H''_{d''}$ are fundamental classes, then $[H'] \otimes [H''] \in H_{d'+d''}$ is a fundamental class for

H. This follows directly from the definitions, and shows in addition that $\text{f-dim}(H' \otimes H'') = \text{f-dim}(H') + \text{f-dim}(H'')$.

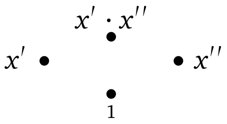
If two Poincaré duality algebras H' and H'' have the same formal dimension d, one can define their **connected sum** $H' \# H''$ in the following way:

$$(H' \# H'')_k = \begin{cases} \mathbb{F} \cdot [H' \# H''] & \text{if } k = d \\ H'_k \oplus H''_k & \text{if } 0 < k < d \\ 1 \cdot \mathbb{F} & \text{if } k = 0. \end{cases}$$

The products of two elements in either H' or H'' are as before, modulo the identification of the three fundamental classes $[H']$, $[H' \# H'']$, $[H'']$. The product of an element of H' of positive degree and of H'' of positive degree is zero. The operation # turns the isomorphism classes of Poincaré duality algebras of a fixed formal dimension d over a fixed ground field $\mathbb{F}$ into a commutative torsion free monoid. The Poincaré duality algebra[1] $H^*(S^d; \mathbb{F})$ serves as a two sided unit: this is the algebra H with $H_0 = \mathbb{F} = H_d$ and all other homogeneous components trivial. Already at this point a number of unsolved problems appear. Here is one such.

PROBLEM I.1.2: *What are the* **indecomposable** *Poincaré duality algebras with respect to the connected sum operation, i.e., what generates the monoid of isomorphism classes of Poincaré duality algebras over a fixed ground field and of a fixed formal dimension under the operation of connected sum? What is the Grothendieck group of this monoid?*

One might hope for a simple answer, such as: the complete intersections provide generators. However this is not the case: the Poincaré duality algebra $H = H^*(S^2 \times S^2; \mathbb{F})$ cannot be written as a nontrivial connected sum. This is best seen on the basis of the accompanying graphic for this algebra.



In the graphic the two generators of the algebra, which appear in degree 2, are x' and x'', whose squares are zero, but whose product is a fundamental class. Up to a change of basis a nontrivial connected sum decomposition $H = H' \# H''$ would have to put x' in H' say, and x'' in H''. But then both $(x')^2$ and $(x'')^2$ would have to be fundamental classes of H' and H'' respectively, so in H the squares of x' and x'' would become a fundamental class, contrary to the fact that they are zero in H.

Moreover complete intersections need not be indecomposable: for example if $\mathbb{CP}(2)$ is the complex projective plane and $\mathbb{CP}(2)\#\mathbb{CP}(2)$ the connected

[1] If X is a topological space $H^*(X; \mathbb{F})$ denotes the cohomology of X with coefficients in $\mathbb{F}$. S^d denotes the d-dimensional sphere.