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Excerpt
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1 Introduction to Viscous Flow

1.1 Why Study Fluid Dynamics?

Fluid dynamics is a branch of classical physics. It is an instance of continuum mechanics. A fluid is a continuous, deformable material. It is a material that flows in response to imposed forces. This is embodied in the everyday experience of draining water from a sink. The water flows under the action of gravity. It does not have a fixed shape; it fills the sink, conforming to its shape. The water flows with variable velocity, depending on its distance from the drain. All these distinguish fluid motion from solid dynamics. As another example, a pump propels water through a pipe or through the cooling system of a car. How does the reciprocating movement of the pump produce directed flow, extending to distant parts of the cooling circuit? One way or the other, the pump must be exerting forces on the fluid; one way or the other, these forces are communicated to distant portions of the fluid and sets them in motion. It is far from obvious what the nature of that flow will be, especially in a complex geometry. It may be laminar, it may be turbulent; it may be unidirectional, it may be recirculating.

Recirculation is the occurrence of backflow, opposite to the direction of the primary stream. This can be seen behind the pedestals supporting a bridge in a swift river. Despite the strong current, the flow direction reverses, and a circulating eddy forms in a region behind the pedestal. How is such behavior understood and predicted? An understanding requires knowledge of viscous action, of vorticity, of turbulence, and of the governing equations.

As a fluid flows around an obstruction, different points in the fluid flow with different velocities. The motion is continuous; hence, the velocity varies smoothly with position. The fluid might be imagined to be divided into infinitesimal volumes – which can be referred to as fluid particles. Then the fluid flow involves relative movement of the particles. Because the flow varies continuously with position, these particles must influence each other to maintain the smoothness of the flow field. That influence is via the forces of pressure and of viscous friction. It is a scientific triumph that when equations are devised to describe these forces acting on the fluid particles, the rich variety of fluid mechanics emerges. That success is most evident in computer simulation of fluid flow.

The trajectories of fluid particles are the streamlines of motion. These streamlines can be curves stretching from inflow to outflow, or they can be closed curves, corresponding to eddying flow. The flow over a rock in a stream or past a spoon stirred through a cup of coffee are examples containing eddying flow. It is not difficult to observe phenomenology of fluid motion: hold a lighted candle, or tissue paper, at arm's length and puff very quickly. A vortex ring travels from your mouth to the candle. There is a time delay before it flickers; that is because the vortex travels with a finite speed. Now suck air into your mouth; the candle shows no evidence of flow. The vortex flows into your mouth. The flow outside is then quite different from when you blew the air out.

Some interesting and challenging scientific and engineering questions already suggest themselves. What causes the eddies and vortices? Why is the outflow so different from the inflow? What drives the flow through a conduit or over an obstacle? How is motion communicated to distant portions of the fluid? How can the flow rate be predicted? How are the streamlines of a fluid flow determined? In any but the simplest cases, these are challenging questions. There is a body of knowledge that can be called on; but one is rapidly struck with how difficult it is to answer even fairly simple questions about fluid flow.

Computer simulation has changed this, rather substantially. Flow in complex geometry can be solved numerically. The phenomena seen in laboratory experiments, and in more casual experience, can be reproduced with quantitative accuracy. To an extent, older theories and analyses have been enlarged by computation. The laws governing fluid flow have been found to be extremely accurate; one wants only sufficient computer power and efficient algorithms to produce solutions. Computer simulation becomes a method of solution, complementary to paper-and-pencil analysis. It provides further understanding of fluid flow and a tool for engineering analysis. However, computer simulations are solutions of a different nature from classical, exact, or approximate solutions. They are numerical data rather than formulas. Traditional theory is not displaced; its role evolves and it provides the understanding needed to formulate and make sense of computer-aided analysis. The additional understanding of fluid dynamics that stems from simulation should be developed in concert with theory. The motive to study fluid dynamics is to understand its phenomena. The approach is to devise and solve the laws of fluid motion.

1.2 Viscosity

At the root is the governing laws. These are the equations of conservation; conservation of mass and momentum, at the present stage. Friction is an important element of fluid flow. In the absence of friction, a flat plate, dragged tangentially across the surface of a tank of water, would slide freely and induce no movement in the water. But, in the presence of friction, motion is communicated to the water adjacent to the plate and thence a circulation is established in the tank.

Friction internal to a fluid flow is characterized by viscosity. The viscosity coefficient, μ, is an empirical property of the fluid. For instance, a liquid is more viscous than a gas. As temperature increases, liquids flow more easily (think of tar, for example) so viscosity decreases with temperature. Gases have the opposite property,

Table 1.1. *Viscosities at room temperature* (20°C)

Fluid	μ g/cm · s	ρ g/cm^3	ν cm^2/s
Air	1.8×10^{-4}	1.2×10^{-3}	0.15
Water	0.01	1.0	0.01
CO_2	1.37×10^{-4}	1.79×10^{-3}	0.077
Engine oil	10	0.89	11.2
Glycerin	7.99	1.26	6.34
Kerosene	0.024	0.78	0.031
Methyl alcohol	0.0055	0.785	0.007

that viscosity increases with temperature. That behavior is less intuitive. It originates in the increased molecular agitation as temperature increases. Clearly, μ must be measured as a function of the fluid and of the temperature. In gases, the coefficient increases approximately as the square root of temperature; in liquids, it falls as the exponential of one over temperature [$\exp(E/kT)$]. Detailed formulas need not be discussed here. Values of μ for many fluids are available in computational fluid dynamics (CFD) codes and in handbooks. The magnitude of the viscosity is essential to determining flow regimes. Table 1.1 contains a few representative values at room temperature.

The coefficient, μ, is the *dynamic* viscosity. For future reference, the *kinematic viscosity* is defined as dynamic viscosity divided by density as follows:

$$\nu = \mu/\rho. \tag{1.1}$$

Kinematic viscosity has dimensions of *length*^{2}/*time*; dynamic viscosity has dimensions of density times this or *mass*/*length* · *time*. Kinematic viscosity is most relevant to constant density, incompressible flow. Oddly enough, the kinematic viscosity of liquids is often lower than that of gasses. For instance, air at 20°C and 1 atmosphere, has a kinematic viscosity of 0.15 cm^2/s; for water it is 0.01 cm^2/s. This is a consequence of the higher density of water.

Viscosity produces forces as a consequence of the relative motion of fluid particles. That might be thought of as friction associated with the particles rubbing across one another. A more correct statement is that viscous *stress* is a consequence of the fluid *rate of strain*.

The need to distinguish rate of strain from simply relative motion is because a fluid in solid body rotation experiences no viscous stress. At a macroscopic level that is clear: if the entire fluid is in solid body rotation, then in a frame rotating with the fluid there is no motion and hence no viscous stress. In a fixed frame, solid body motion means that the velocity is Ωr in the angular direction. There is relative motion in the sense that fluid at $r = 0$ is at rest, whereas at $r > 0$ it is in motion, but there is no viscous stress.

The same concept applies, less obviously, at any point in a nonrotating fluid. Relative motion can be separated into rotation and rate of strain; only the latter produces viscous stress. The velocity is a field: at any point $\boldsymbol{x} = (x, y, z)$, three components of velocity (u, v, w) can be measured. Rate of strain is a measure of how this velocity varies from point to point within the vicinity of $\boldsymbol{x}$. Mathematically, if two points are separated by a distance $d\boldsymbol{x}$, their relative velocity is

$$u_i(\boldsymbol{x} + d\boldsymbol{x}) - u_i(\boldsymbol{x}) \approx d\boldsymbol{x} \cdot \nabla u_i,$$

where $i = 1, 2$, or 3 corresponding to u, v, or w. The last term expands to

$$dx\frac{\partial u_i}{\partial x} + dy\frac{\partial u_i}{\partial y} + dz\frac{\partial u_i}{\partial z}. \tag{1.2}$$

The convention of summation on repeated subscripts permits this to be written equivalently as

$$dx_j\frac{\partial u_i}{\partial x_j}.$$

Because the same index, j, appears twice in this product, the convention is that j is summed from 1 through 3, so that this is exactly the same expression as Eq. (1.2). This is a rather terse introduction to index notation. The uninitiated reader might want to write out corresponding formulas in index notation and in Cartesian components. That exercise is illustrated in the next paragraph.

We now introduce the separation of the velocity gradient into rate of strain and rotation; it is equivalent to a separation into symmetric and antisymmetric components, respectively. Specifically,

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{1.3}$$

The first term on the right is the rate of strain, which can be denoted S_{ij}; the second is minus the rate of rotation, which can be denoted $-\Omega_{ij}$:

$$\begin{aligned} S_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \\ -\Omega_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \end{aligned} \tag{1.4}$$

Only S_{ij} produces viscous stress. In the standard index notation used here, i and j are dummy subscripts, for which any of the numbers (1, 2, 3) corresponding to the directions (x, y, z) can be substituted. For instance, with $i = 1$ and $j = 2$, Eq. (1.3) says

$$\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right).$$

In solid body rotation, $u = -\Omega y$ and $v = \Omega x$. The first term vanishes and the second equals $-\Omega$; the rate of strain is zero under solid body rotation. An irrotational flow is one for which the second term vanishes: $\partial u/\partial y = \partial v/\partial x$. For instance, $u = \alpha y$, $v = \alpha x$ is an irrotational straining field.

The other element in the description of viscous forces is that they are characterized as a *stress* rather than as a *force* per se. This should not be unfamiliar: pressure also is a stress. That is, it is a force per unit area, acting in the direction normal to a surface. Viscous stress is similar: it is a force per unit area, but it need not act normal to the surface; it can have both normal and tangential components. If the fluid motion is just a shearing tangential to a surface, then it is clear that the viscous stress will cause a force in the tangential direction. It is less obvious, but true, that if the fluid motion is toward the surface there will also be a component of force in the normal direction. That is best described mathematically, as will be done in the next section.

Given the stress, the corresponding force on an object is obtained by integrating it over the entire area of the surface. This can be accomplished inside a CFD code, so one need only understand the origin of the force that is being computed. The reason that the force originates as a stress is that fluids are deformable material, so their dynamical properties must be defined for the fluid particles. Stress is the force per area acting on a fluid particle. It is independent of the size of the infinitesimal fluid particle. The force, by contrast, is proportional to the size. In other words, $force = stress \cdot area$. As the area becomes tiny, the stress remains finite and the force becomes tiny. Just like pressure, viscous stress is the quantity that is defined pointwise throughout the fluid. Unlike pressure, it exists only in a flowing fluid.

Viscous stress is incorporated into the governing, Navier–Stokes, momentum equations that are solved by CFD software. The gist of those equations is the next topic.

1.3 Navier–Stokes Equations

It is assumed that the reader has studied elementary fluid mechanics and has been exposed to the basic notions of fluid flow. These include the role of pressure in momentum transport, conservation of mass in an incompressible, deformable fluid medium, and the origin of viscous, frictional forces. The last have just been discussed. This section provides an informal description of the Navier–Stokes momentum equation for constant density, incompressible flow. Because we rely on CFD software for solutions to the equations, we will abbreviate the treatment that can be found more fully in standard texts on viscous flow (White, 1991).

The Navier–Stokes equations were named for the French engineer and scientist Claude Louis Marie Henri Navier and the English mathematical physicist George Gabriel Stokes. The essential form of these equations was set forth by Navier in 1822; however, he did not properly treat the origin of viscous stress. The latter was addressed by others, in particular Poisson and Saint-Venant, but independently developed by Stokes in 1845. Stokes constructed a number of solutions to the equations of viscous flow, which confirmed their ability to describe fluid dynamical phenomena. An example is creeping flow, also called “Stokes flow,” which we discuss in

Chapter 3. Navier is properly credited for the seminal formulation of the Navier–Stokes equations and Stokes for ushering their entry into theoretical physics.

Essentially, the Navier–Stokes momentum equation is an expression of Newton's law $m\boldsymbol{a} = \boldsymbol{F}$ applied to an infinitesimal fluid volume. Here we use the convention that bold letters denote vectors. On a volumetric basis, the mass becomes mass per unit volume or density ρ. The acceleration becomes that following the fluid element, or the convective derivative of velocity, $D\boldsymbol{u}/Dt$, and the force per unit volume includes both pressure and viscous contributions, as follows:

$$\rho \frac{D\boldsymbol{u}}{Dt} = \mathcal{F}_{\text{press}} + \mathcal{F}_{\text{viscous}}. \tag{1.5}$$

We must flesh out the meaning of these various terms.

The equations of motion referred to fluid particles is called the *Lagrangian* description. The fluid particle occupies a position $\boldsymbol{X}(t;\boldsymbol{x}_0)$ that changes with time. The particle is labeled here by its initial location $\boldsymbol{x}_0$. It is more convenient to describe the flow in terms of the velocity at fixed points. We think of a flow field, $\boldsymbol{u}(\boldsymbol{x})$, rather than of the dynamics of particles. The only complication in applying Newton's law to the field is transforming the acceleration of the fluid particle into velocity changes at a fixed position.

To derive the requisite expression, first consider a material that is carried with the fluid element. The material has a concentration c. An observer at a fixed point, $\boldsymbol{x}$, will see the concentration change as different particles arrive. At any given time, the concentration is that of the particle currently at $\boldsymbol{x}$, that is, of the particle with $\boldsymbol{X}(t) = \boldsymbol{x}$. At time δt later, a particle that was at $\boldsymbol{X} - \delta\boldsymbol{X}$, say, will have moved to $\boldsymbol{x}$. The observer then sees its concentration $c(\boldsymbol{X} - \delta\boldsymbol{X})$. Thus the observer sees the change

$$\frac{\partial c}{\partial t} = \frac{c(\boldsymbol{X} - \delta\boldsymbol{X}) - c(\boldsymbol{X})}{\delta t} \approx \frac{-\delta\boldsymbol{X} \cdot \nabla c}{\delta t}$$

as the fluid element occupying position $\boldsymbol{x}$ changes from that at time t to that at time $t + \delta t$. $\delta\boldsymbol{X}/\delta t$ is the velocity $\boldsymbol{u}$. Hence, the motion of fluid elements produces the time variation

$$\frac{\partial c}{\partial t} = -\boldsymbol{u} \cdot \nabla c.$$

There is nothing special about concentration: the same result applies to any quantity convected with the flow. Putting both terms on the same side of the last equation shows that a transported quantity satisfies

$$\frac{Dc}{Dt} \equiv \frac{\partial c}{\partial t} + \boldsymbol{u} \cdot \nabla c = 0.$$

This is a statement that the changes at a fixed position are simply due to different elements arriving at that position, carrying their particular concentration. If the quantity were not simply convected but also underwent some change, then the right side would be nonzero.

Thus, we arrive at the expression of Newton's law at a fixed point – which is called the *Eulerian* description. The quantity being carried is now the fluid momentum, $\rho\boldsymbol{u}$. It is carried with the particles but also changes as a consequence of forces. The flow field obeys

$$\rho\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] = \mathcal{F}_{\text{press}} + \mathcal{F}_{\text{viscous}}. \tag{1.6}$$

A more general concept than the right side of Eqs. (1.5) or (1.6) is to combine pressure and viscous terms into a single stress tensor. Stresses are forces acting on a surface per unit area. *Pressure* times *area* is a force acting perpendicularly to the surface; in other words, it is a *normal stress*. Viscosity produces both *normal* and *tangential*, or shearing, stresses. The tangential stress is quite intuitive: it is analogous to the force felt when rubbing one hand over the other. As mentioned in the last section, it is also the case that viscosity produces a component of normal force, parallel to pressure. On any surface, the aggregate stress is a vector, with components both normal and tangential to the surface, composed of contributions from pressure and from viscosity.

How are forces produced by stress represented? The aggregate force can be denoted $\boldsymbol{F}_s$. It is the force produced by a stress acting on a surface. Consider that surface to be a small, differential area, dA. Further, let that area be the magnitude of a vector $d\boldsymbol{A}$ that is directed normal to the surface. The stress, $\boldsymbol{\sigma}$, now can be defined: the force is the dot product of the stress with the area vector

$$\boldsymbol{F}_s = \boldsymbol{\sigma}\cdot d\boldsymbol{A}. \tag{1.7}$$

This simply defines the *stress tensor* $\boldsymbol{\sigma}$ as a matrix relating the force vector to the area vector. Purely as a matter of consistency, Eq. (1.7) shows that stress has dimensions of force per area. In component form, the matrix relation [Eq. (1.7)] between stress and surface force is stated as

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \cdot \begin{pmatrix} dA_x \\ dA_y \\ dA_z \end{pmatrix}. \tag{1.8}$$

The term stress *tensor* was slipped into the text, above. A tensor is a generalization of a vector. A vector has a direction. A tensor has one or more directions. A vector is a first-order tensor, having a single direction. Stress is a second-order tensor; it is associated with two directions. The two directions are that of the force vector and that of the area vector. Matrix $\boldsymbol{\sigma}$ relates the direction of a force acting on a surface to the area vector of that surface.

Equation (1.7), integrated over a solid surface, gives the net force exerted by the flowing fluid. When, in later chapters, we consider examples of fluid forces on objects, the quantity

$$\int \boldsymbol{\sigma}\cdot d\boldsymbol{A} \tag{1.9}$$

is being evaluated by integration over the entire surface in question.

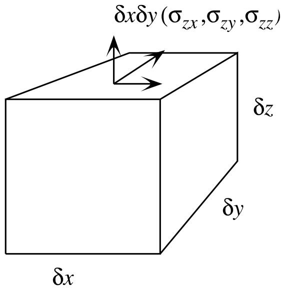

Figure 1.1. Stresses on a face of a fluid element.

The force evaluation usually can be obtained with the CFD software; but it should be understood that the computer code is providing this surface integral of the stress tensor. In fact the force can be broken down into contributions from pressure and viscosity. The accuracy of the numerical evaluations depends on how finely the mesh covers the surface and on how accurately the viscous and pressure stresses are computed by the flow solution. The force evaluation is a postprocessing of that solution.

The presence of two contributions to stress can be acknowledged by writing

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{viscous}} + \boldsymbol{\sigma}_{\text{pressure}}, \tag{1.10}$$

Precise expressions for these two contributions will be given shortly.

Forces due to stress also act inside the fluid. Within the fluid, they are the force on the face of an infinitesimal fluid element (Figure 1.1). That element will move if there is an imbalance of forces. Consider two opposite sides of an element having oppositely directed normals. The force imbalance is due to a difference between the stresses acting on the opposite sides, that is, to a differential of stress. If $\sigma_L A$ is the force on the left side of the fluid element and $\sigma_R A$ is the force on its right side, then $(\sigma_R - \sigma_L)A$ is the force imbalance. If ℓ is the length of the element and $\mathcal{V} = A\ell$ its volume, then the resultant force is

$$\frac{\sigma_R - \sigma_L}{\ell} A\ell = \frac{\sigma_R - \sigma_L}{\ell}\mathcal{V} \approx \frac{d\sigma}{d\ell}\mathcal{V}.$$

That is, the force per unit volume is the directional derivative of the stress. The equation of motion is now $\mathcal{M}\, Du/Dt = \mathcal{V}\, d\sigma/d\ell$. The ratio $\mathcal{M}/\mathcal{V}$ is the density ρ. It takes only a bit of elaboration to recognize that the directional derivative $d\sigma/d\ell$ should be generalized to the divergence of the stress $\nabla \cdot \boldsymbol{\sigma}$. Essentially, the gradient operator gives the directional derivative.

Hence, the force that appears in the Navier–Stokes momentum equation is the divergence of stress, and Newton's law becomes

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right] = \nabla \cdot \boldsymbol{\sigma}. \tag{1.11}$$

This is simply Newton's law (1.5) when the force is caused by a stress gradient. As explained above, $\partial \boldsymbol{u}/\partial t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ is the Eulerian form for the acceleration $D\boldsymbol{u}/Dt$.

Now to make explicit the observation that stress is composed of a part due to pressure and a part due to viscosity. The part due to pressure is just pressure times the identity matrix, with a minus sign:

$$\boldsymbol{\sigma}_{\text{press}} = -p\boldsymbol{I}. \tag{1.12}$$

The minus sign arises because pressure applied to a surface acts inward to the surface. For instance, if the surface is the xy plane, an imposed high pressure will push down in the $-z$ direction on the surface. That is a special case of the general formula

$$\boldsymbol{F}_{\text{press}} = \boldsymbol{\sigma}_{\text{press}} \cdot d\boldsymbol{A} = -p\boldsymbol{I} \cdot d\boldsymbol{A} = -pd\boldsymbol{A}.$$

Pressure times *area* is the inward force. The pressure contribution to stress gives rise to a pressure gradient on the right side of Eq. (1.11): $\nabla \cdot \boldsymbol{\sigma}_{\text{press}} = -\nabla p$. This might be comforting to the reader who is wondering why there is no pressure gradient in Eq. (1.11).

The representation of viscous stress is less obvious. In a Newtonian fluid, it is assumed proportional to the rate of strain of the fluid motion, as described in §1.2. Viscosity is simply the coefficient of proportionality. This is stated as

$$\boldsymbol{\sigma}_{\text{viscous}} = \mu(\nabla\boldsymbol{u} + {}^t[\nabla\boldsymbol{u}]), \tag{1.13}$$

where $\nabla\boldsymbol{u}$ is a matrix of velocity derivatives and ${}^t[\nabla\boldsymbol{u}]$ is its transpose. Equation (1.13) is a statement in vector form of the relation

$$\sigma_{ij} = 2\mu S_{ij}$$

in index form, with S_{ij} given by Eq. (1.4). When this formula is used for an incompressible fluid, the viscous force simplifies to the Laplacian of velocity: $\nabla \cdot \boldsymbol{\sigma}_{\text{viscous}} = \mu\nabla^2\boldsymbol{u}$. The Navier–Stokes equations of an incompressible (Newtonian) fluid assume the form

$$\rho\left[\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] = -\nabla p + \mu\nabla^2\boldsymbol{u}. \tag{1.14}$$

The pressure and viscous forces are stated explicitly here. Lengthy discussions of the derivation of this equation, and some caveats that must be made in our derivation, can be found in standard texts (Panton, 1997; White, 1991). In fact, it is an equation that is quite remarkable for its ability to describe the phenomena of fluid flow. It is equally remarkable for its mathematical intransigence. Basically, it is the momentum equation that is solved by CFD software.

That is not quite correct: (1.14) is an equation for the velocity vector $\boldsymbol{u}$ or for the u, v, and w components of velocity. It contains the further fluid properties ρ and p. Consider constant density flow, such as air at low speed or water without significant contaminant concentrations. Then pressure is the only additional dynamical variable. Another equation is needed to predict it.

In many situations, the interest is in essentially incompressible flow. Incompressible means that pressure changes produce negligible density changes: $d\rho/dp \approx 0$. The reader might recognize that $d\rho/dp$ is one over the squared sound speed, c^{-2}, in a gas. In that case, the approximation of incompressibility is justified if the Mach number, $M = u/c$, is small. In other words, the smallness of $d\rho/dp$ is a relative statement. It says that pressure variations go primarily into accelerating the flow rather than into changing the density. Certainly air is compressible; it can be compressed in a pump, or into a tire. But for the purpose of fluid dynamics, it can be treated as incompressible if the flowing fluid does not cause significant compression. This is normally the case when the Mach number is low. Liquids are almost always incompressible. Density variations can occur, such as those due to salt dissolved in water, but, again, it is not the fluid velocity that causes density to vary.

The condition of incompressibility is that infinitesimal fluid elements retain their volume. Their shape will deform, but the net volume associated with an element is constant. We are defining the fluid element as a fixed amount of mass, so that constant density is equivalent to constant volume. The condition of incompressibility requires the divergence of the velocity to vanish:

$$\nabla \cdot u = 0 \tag{1.15}$$

or $\partial_x u + \partial_y v + \partial_z w = 0$. Essentially, this is saying that the volume deformations in the x, y, and z directions sum to zero. It may be justified as follows.

Consider a rectangular material element – for instruction, we work in two dimensions. The corners of the rectangle move with the fluid. Let the lower-left and upper-right corners be (X, Y) and $(X + \delta X, Y + \delta Y)$. The area is $\mathcal{A} = \delta X \delta Y$. The rectangle will deform as the fluid flows, but if it is incompressible, the area does not change. Then Eq. (1.15) follows from differentiating the area with respect to time and setting it to 0:

$$\begin{aligned}\frac{d\mathcal{A}}{dt} = 0 &= \delta Y \frac{d}{dt}\delta X + \delta X \frac{d}{dt}\delta Y \\ &= \mathcal{A}\left(\frac{1}{\delta X}\frac{d}{dt}\delta X + \frac{1}{\delta Y}\frac{d}{dt}\delta Y\right) \\ &= \mathcal{A}\left(\frac{\delta u}{\delta X} + \frac{\delta v}{\delta Y}\right) = \mathcal{A}\nabla \cdot \boldsymbol{u},\end{aligned}$$

where $\delta u = d\delta X/dt$ and $\delta v = d\delta Y/dt$ were substituted. In three dimensions, the same argument is applied to a fluid volume.

1.4 Reynolds Number

The boundary condition on rigid surfaces is that the fluid immediately adjacent to the surface moves with the wall velocity. This is the no-slip condition. It says that there is no discontinuity in velocity at the wall. As a stationary wall is approached, the fluid velocity tends continuously to zero. This no-slip boundary condition is not indisputable. In rarified gases, and at the intersection of a gas–fluid interface with a wall, it is violated. But at normal densities and pressures, and in homogeneous