1

The intriguing natural numbers

'The time has come,' the Walrus said, 'To talk of many things.'
Lewis Carroll

1.1 Polygonal numbers

We begin the study of elementary number theory by considering a few basic properties of the set of natural or counting numbers, $\{1, 2, 3, \ldots\}$. The natural numbers are closed under the binary operations of addition and multiplication. That is, the sum and product of two natural numbers are also natural numbers. In addition, the natural numbers are commutative, associative, and distributive under addition and multiplication. That is, for any natural numbers, a, b, c:

$$
\begin{array}{lll}
a + (b + c) = (a + b) + c, & a(bc) = (ab)c & \text{(associativity);} \\
a + b = b + a, & ab = ba & \text{(commutativity);} \\
a(b + c) = ab + ac, & (a + b)c = ac + bc & \text{(distributivity).}
\end{array}
$$

We use juxtaposition, xy, a convention introduced by the English mathematician Thomas Harriot in the early seventeenth century, to denote the product of the two numbers x and y. Harriot was also the first to employ the symbols '$>$' and '$<$' to represent, respectively, 'is greater than' and 'is less than'. He is one of the more interesting characters in the history of mathematics. Harriot traveled with Sir Walter Raleigh to North Carolina in 1585 and was imprisoned in 1605 with Raleigh in the Tower of London after the Gunpowder Plot. In 1609, he made telescopic observations and drawings of the Moon a month before Galileo sketched the lunar image in its various phases.

One of the earliest subsets of natural numbers recognized by ancient mathematicians was the set of polygonal numbers. Such numbers represent an ancient link between geometry and number theory. Their origin can be traced back to the Greeks, where properties of oblong, triangular, and square numbers were investigated and discussed by the sixth century BC, pre-Socratic philosopher Pythagoras of Samos and his followers. The

Greeks established the deductive method of reasoning whereby conclusions are derived using previously established results.

At age 18, Pythagoras won a prize for wrestling at the Olympic games. He studied with Thales, father of Greek mathematics, traveled extensively in Egypt and was well acquainted with Babylonian mathematics. At age 40, after teaching in Elis and Sparta, he migrated to Magna Graecia, where the Pythagorean School flourished at Croton in what is now Southern Italy. The Pythagoreans are best known for their theory of the transmigration of souls and their belief that numbers constitute the nature of all things. The Pythagoreans occupied much of their time with mysticism and numerology and were among the first to depict polygonal numbers as arrangements of points in regular geometric patterns. In practice, they probably used pebbles to illustrate the patterns and in doing so derived several fundamental properties of polygonal numbers. Unfortunately, it was their obsession with the deification of numbers and collusion with astrologers that later prompted Saint Augustine to equate mathematicans with those full of empty prophecies who would willfully sell their souls to the Devil to gain the advantage.

The most elementary class of polygonal numbers described by the early Pythagoreans was that of the oblong numbers. The nth oblong number, denoted by o_n, is given by $n(n+1)$ and represents the number of points in a rectangular array having n columns and $n+1$ rows. Diagrams for the first four oblong numbers, 2, 6, 12, and 20, are illustrated in Figure 1.1.

The triangular numbers, 1, 3, 6, 10, 15, $\ldots$, t_n, $\ldots$, where t_n denotes the nth triangular number, represent the numbers of points used to portray equilateral triangular patterns as shown in Figure 1.2. In general, from the sequence of dots in the rows of the triangles in Figure 1.2, it follows that t_n, for $n \geqslant 1$, represents successive partial sums of the first n natural numbers. For example, $t_4 = 1+2+3+4 = 10$. Since the natural numbers are commutative and associative,

$$t_n = 1 + 2 + \cdots + (n-1) + n$$

and

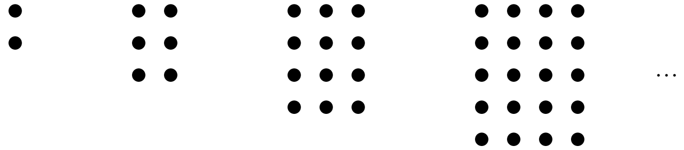

Figure 1.1

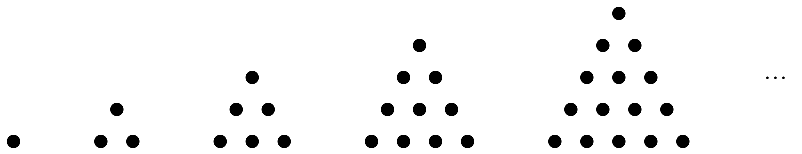

Figure 1.2

$$t_n = n + (n-1) + \cdots + 2 + 1;$$

adding columnwise, it follows that $2t_n = (n+1) + (n+1) + \cdots + (n+1) = n(n+1)$. Hence, $t_n = n(n+1)/2$. Multiplying both sides of the latter equation by 2, we find that twice a triangular number is an oblong number. That is, $2t_n = o_n$, for any positive integer n. This result is illustrated in Figure 1.3 for the case when $n = 6$. Since $2 + 4 + \cdots + 2n = 2(1 + 2 + \cdots + n) = 2 \cdot n(n+1)/2 = n(n+1) = o_n$, the sum of the first n even numbers equals the nth oblong number.

The square numbers, 1, 4, 9, 16, ..., were represented geometrically by the Pythagoreans as square arrays of points, as shown in Figure 1.4. In 1225, Leonardo of Pisa, more commonly known as Fibonacci, remarked, in *Liber quadratorum* (*The Book of Squares*) that the nth square number, denoted by s_n, exceeded its predecessor, s_{n-1}, by the sum of the two roots. That is $s_n = s_{n-1} + \sqrt{s_n} + \sqrt{s_{n-1}}$ or, equivalently, $n^2 = (n-1)^2 + n + (n-1)$. Fibonacci, later associated with the court of Frederick II, Emperor of the Holy Roman Empire, learned to calculate with Hindu–Arabic

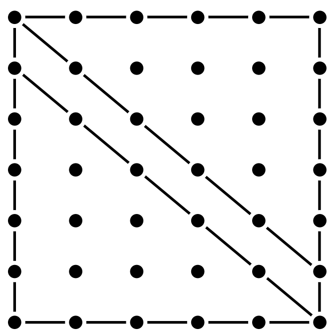

Figure 1.3

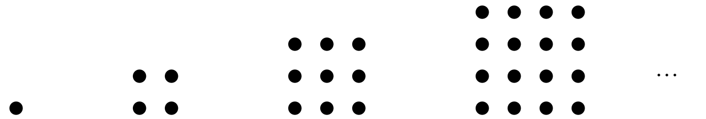

Figure 1.4

numerals while in Bougie, Algeria, where his father was a customs officer. He was a direct successor to the Arabic mathematical school and his work helped popularize the Hindu–Arabic numeral system in Europe. The origin of Leonardo of Pisa's sobriquet is a mystery, but according to some sources, Leonardo was figlio de (son of) Bonacci and thus known to us patronymically as Fibonacci.

The Pythagoreans realized that the nth square number is the sum of the first n odd numbers. That is, $n^2 = 1 + 3 + 5 + \cdots + (2n - 1)$, for any positive integer n. This property of the natural numbers first appears in Europe in Fibonacci's *Liber quadratorum* and is illustrated in Figure 1.5, for the case when $n = 6$.

Another interesting property, known to the early Pythagoreans, appears in Plutarch's *Platonic Questions*. Plutarch, a second century Greek biographer of noble Greeks and Romans, states that eight times a triangular number plus one is square. Using modern notation, we have $8t_n + 1 = 8[n(n + 1)/2] + 1 = (2n + 1)^2 = s_{2n+1}$. In Figure 1.6, the result is illustrated for the case $n = 3$. It is in Plutarch's biography of Marcellus that we find one of the few accounts of the death of Archimedes during the siege of Syracuse, in 212 BC.

Around the second century BC, Hypsicles [HIP sih cleez], author of

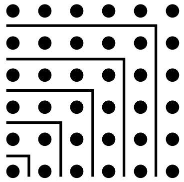

Figure 1.5

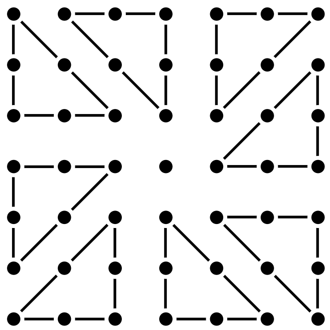

Figure 1.6

Book XIV, a supplement to Book XIII of Euclid's *Elements* on regular polyhedra, introduced the term polygonal number to denote those natural numbers that were oblong, triangular, square, and so forth. Earlier, the fourth century BC philosopher Plato, continuing the Pythagorean tradition, founded a school of philosophy near Athens in an area that had been dedicated to the mythical hero Academus. Plato's Academy was not primarily a place for instruction or research, but a center for inquiry, dialogue, and the pursuit of intellectual pleasure. Plato's writings contain numerous mathematical references and classification schemes for numbers. He firmly believed that a country's leaders should be well-grounded in Greek arithmetic, that is, in the abstract properties of numbers rather than in numerical calculations. Prominently displayed at the Academy was a maxim to the effect that none should enter (and presumably leave) the school ignorant of mathematics. The epigram appears on the logo of the American Mathematical Society. Plato's Academy lasted for nine centuries until, along with other pagan schools, it was closed by the Byzantine Emperor Justinian in 529.

Two significant number theoretic works survive from the early second century, *On Mathematical Matters Useful for Reading Plato* by Theon of Smyrna and *Introduction to Arithmetic* by Nicomachus [nih COM uh kus] of Gerasa. Smyrna in Asia Minor, now Izmir in Turkey, is located about 75 kilometers northeast of Samos. Gerasa, now Jerash in Jordan, is situated about 25 kilometers north of Amman. Both works are philosophical in nature and were written chiefly to clarify the mathematical principles found in Plato's works. In the process, both authors attempt to summarize the accumulated knowledge of Greek arithmetic and, as a consequence, neither work is very original. Both treatises contain numerous observations concerning polygonal numbers; however, each is devoid of any form of rigorous proofs as found in Euclid. Theon's goal was to describe the beauty of the interrelationships between mathematics, music, and astronomy. Theon's work contains more topics and was a far superior work mathematically than the *Introduction*, but it was not as popular. Both authors note that any square number is the sum of two consecutive triangular numbers, that is, in modern notation, $s_n = t_n + t_{n-1}$, for any natural number $n > 1$. Theon demonstrates the result geometrically by drawing a line just above and parallel to the main diagonal of a square array. For example, the case where $n = 5$ is illustrated in Figure 1.7. Nicomachus notes that if the square and oblong numbers are written alternately, as shown in Figure 1.8, and combined in pairs, the triangular numbers are produced. That is, using modern notation, $t_{2n} = s_n + o_n$ and $t_{2n+1} = s_{n+1} + o_n$, for any natural

Table 1.1.

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

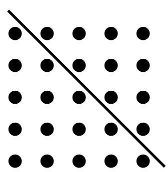

Figure 1.7

$$
\begin{array}{ccccccccccccccccccc}
s_1 & & o_1 & & s_2 & & o_2 & & s_3 & & o_3 & & s_4 & & o_4 & & s_5 & & o_5 \\
1 & & 2 & & 4 & & 6 & & 9 & & 12 & & 16 & & 20 & & 25 & & 30 \\
 & 3 & & 6 & & 10 & & 15 & & 21 & & 28 & & 36 & & 45 & & 55 & \\
 & t_2 & & t_3 & & t_4 & & t_5 & & t_6 & & t_7 & & t_8 & & t_9 & & t_{10} &
\end{array}
$$

Figure 1.8

number n. From a standard multiplication table of the first ten natural numbers, shown in Table 1.1, Nicomachus notices that the major diagonal is composed of the square numbers and the successive squares s_n and s_{n+1} are flanked by the oblong numbers o_n. From this, he deduces two properties that we express in modern notation as $s_n + s_{n+1} + 2o_n = s_{2n+1}$ and $o_{n-1} + o_n + 2s_n = s_{2n}$.

Nicomachus extends his discussion of square numbers to the higher dimensional cubic numbers, 1, 8, 27, 64, ..., and notes, but does not establish, a remarkable property of the odd natural numbers and the cubic numbers illustrated in the triangular array shown in Figure 1.9, namely, that the sum of the nth row of the array is n^3. It may well have been Nicomachus's only original contribution to mathematics.

1	1
3 5	8
7 9 11	27
13 15 17 19	64
21 23 25 27 29	125
....	

Figure 1.9

In the *Introduction*, Nicomachus discusses properties of arithmetic, geometric, and harmonic progressions. With respect to the arithmetic progression of three natural numbers, he observes that the product of the extremes differs from the square of the mean by the square of the common difference. According to this property, known as the *Regula Nicomachi*, if the three terms in the progression are given by $a - k$, a, $a + k$, then $(a - k)(a + k) + k^2 = a^2$. In the Middle Ages, rules for multiplying two numbers were rather complex. The Rule of Nicomachus was useful in squaring numbers. For example, applying the rule for the case when $a = 98$, we obtain $98^2 = (98 - 2)(98 + 2) + 2^2 = 96 \cdot 100 + 4 = 9604$.

After listing several properties of oblong, triangular, and square numbers, Nicomachus and Theon discuss properties of pentagonal and hexagonal numbers. Pentagonal numbers, 1, 5, 12, 22, ..., ${p^5}_n$, ..., where ${p^5}_n$ denotes the nth pentagonal number, represent the number of points used to construct the regular geometric patterns shown in Figure 1.10. Nicomachus generalizes to heptagonal and octagonal numbers, and remarks on the patterns that arise from taking differences of successive triangular, square, pentagonal, heptagonal, and octagonal numbers. From this knowledge, a general formula for polygonal numbers can be derived. A practical technique for accomplishing this involving successive differences appeared in a late thirteenth century Chinese text *Works and Days Calendar* by Wang Xun (SHUN) and Guo Shoujing (GOW SHOE GIN). The method was mentioned in greater detail in 1302 in *Precious Mirror of the Four*

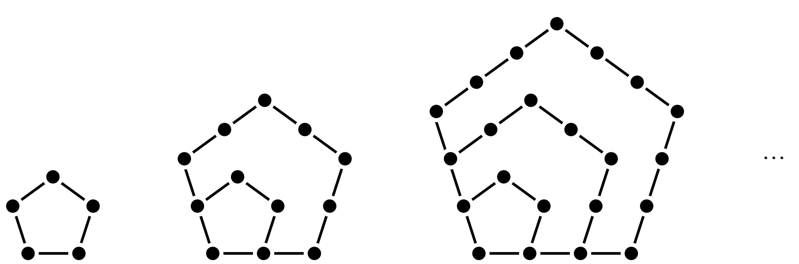

Figure 1.10

Elements by Zhu Shijie (ZOO SHE GEE), a wandering scholar who earned his living teaching mathematics. The method of finite differences was rediscovered independently in the seventeenth century by the British mathematicians Thomas Harriot, James Gregory, and Isaac Newton.

Given a sequence, a_k, a_{k+1}, a_{k+2}, ..., of natural numbers whose rth differences are constant, the method yields a polynomial of degree $r-1$ representing the general term of the given sequence. Consider the binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \text{for } 0 \leqslant k \leqslant n, \quad \binom{n}{0} = 1, \quad \text{and otherwise } \binom{n}{k} = 0,$$

where for any natural number n, n factorial, written $n!$, represents the product $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ and, for consistency, $0! = 1$. The exclamation point used to represent factorials was introduced by Christian Kramp in 1802. The numbers, $\binom{n}{k}$, are called the binomial coefficients because of the role they play in the expansion of $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$. For example,

$$\begin{aligned}(a+b)^3 &= \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3.\end{aligned}$$

Denote the ith differences, Δ_i, of the sequence a_k, a_{k+1}, a_{k+2}, ... by d_{i1}, d_{i2}, d_{i3}, ..., and generate the following finite difference array:

$$\begin{array}{ccccccccccccccc}
n & k & & k+1 & & k+2 & & k+3 & & k+4 & & k+5 & & k+6 \\
a_n & a_k & & a_{k+1} & & a_{k+2} & & a_{k+3} & & a_{k+4} & & a_{k+5} & & a_{k+6} \\
\Delta_1 & & d_{11} & & d_{12} & & d_{13} & & d_{14} & & d_{15} & & d_{16} & \\
\Delta_2 & & & d_{21} & & d_{22} & & d_{23} & & d_{24} & & d_{25} & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\Delta_r & & & & d_{r1} & & d_{r2} & & d_{r3} & & d_{r4} & & &
\end{array}$$

If the rth differences d_{r1}, d_{r2}, d_{r3}, ... are equal, then working backwards and using terms in the leading diagonal each term of the sequence a_k, a_{k+1}, a_{k+2}, ... can be determined. More precisely, the finite difference array for the sequence $b_n = \binom{n-k}{m}$, for $m = 0, 1, 2, 3, \ldots, r$, $n = k, k+1, k+2, \ldots$, and a fixed value of k, has the property that the mth differences, Δ_m, consist of all ones and, except for $d_{m1} = 1$ for $1 \leqslant m \leqslant r$, the leading diagonal is all zeros. For example, if $m = 0$, the finite difference array for $a_n = \binom{n-k}{0}$ is given by

$$\begin{array}{cccccccccccccc}
n & k & & k+1 & & k+2 & & k+3 & & k+4 & & k+5 & & k+6 \\
b_n & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
\Delta_1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

If $m = 1$, the finite difference array for $a_n = \binom{n-k}{1}$ is given by

n	k		$k+1$		$k+2$		$k+3$		$k+4$		$k+5$		$k+6$
b_n	0		1		2		3		4		5		6
Δ_1		1		1		1		1		1		1	
Δ_2			0		0		0		0		0		0

If $m = 2$, the finite difference array for $a_n = \binom{n-k}{2}$ is given by

n	k		$k+1$		$k+2$		$k+3$		$k+4$		$k+5$		$k+6$
b_n	0		0		1		3		6		10		15
Δ_1		0		1		2		3		4		5	
Δ_2			1		1		1		1		1		1
Δ_3				0		0		0		0		0	

The leading diagonals of the finite difference array for the sequence a_k, $a_{k+1}, a_{k+2}, \ldots,$ and the array defined by

$$a_k\binom{n-k}{0} + d_{11}\binom{n-k}{1} + d_{21}\binom{n-k}{2} + \cdots + d_{r1}\binom{n-k}{r}$$

are identical. Therefore,

$$a_n = a_k\binom{n-k}{0} + d_{11}\binom{n-k}{1} + d_{21}\binom{n-k}{2} + \cdots + d_{r1}\binom{n-k}{r},$$

for $n = k, k+1, k+2, \ldots$.

Example 1.1 The finite difference array for the pentagonal numbers, 1, 5, 12, 22, 35, $\ldots$, $p^5{}_n$, $\ldots$ is given by

n	1		2		3		4		5		6		$\ldots$
$p^5{}_n$	1		5		12		22		35		51		$\ldots$
Δ_1		4		7		10		13		16		$\ldots$	
Δ_2			3		3		3		3		$\ldots$		

Our indexing begins with $k = 1$. Therefore

$$p^5{}_n = 1\cdot\binom{n-1}{0} + 4\cdot\binom{n-1}{1} + 3\cdot\binom{n-1}{2} = 1 + 4(n-1) + 3\frac{(n-1)(n-2)}{2}$$
$$= \frac{3n^2 - n}{2}.$$

A more convenient way to determine the general term of sequences with finite differences is the following. Since the second differences of the pentagonal numbers sequence is constant, consider the sequence whose general term is $f(n) = an^2 + bn + c$, whose first few terms are given by $f(1) = a + b + c$, $f(2) = 4a + 2b + c$, $f(3) = 9a + 3b + c$, $f(4) = 16a + 4b + c$, and whose finite differences are given by

$$\begin{array}{llll} a+b+c & 4a+2b+c & 9a+3b+c & 16a+4b+c\ldots \\ & 3a+b & 5a+b & 7a+b\ldots \\ & 2a & 2a & \ldots \end{array}$$

Matching terms on the first diagonal of the pentagonal differences with those of $f(n)$ yields

$$2a = 3$$

$$3a + b = 4$$

$$a + b + c = 1.$$

Hence, $a = \frac{3}{2}$, $b = -\frac{1}{2}$, $c = 0$, and $f(n) = \frac{3}{2}n^2 - \frac{1}{2}n$.

From Table 1.2, Nicomachus infers that the sum of the nth square and the $(n-1)$st triangular number equals the nth pentagonal number, that is, for any positive integer n, $p^5{}_n = s_n + t_{n-1}$. For example, if $n = 6$, $s_6 + t_5 = 36 + 15 = 51 = p^5{}_6$. He also deduces from Table 1.2 that three times the $(n-1)$st triangular number plus n equals the nth pentagonal number. For example, for $n = 9$, $3 \cdot t_8 + 9 = 3 \cdot 36 + 9 = 117 = p^5{}_9$.

In general, the m-gonal numbers, for $m = 3, 4, 5, \ldots,$ where m refers to the number of sides or angles of the polygon in question, are given by the sequence of numbers whose first two terms are 1 and m and whose second common differences equal $m - 2$. Using the finite difference method outlined previously we find that $p^m{}_n = (m-2)n^2/2 - (m-4)n/2$, where $p^m{}_n$ denotes the nth m-gonal number. Triangular numbers correspond to 3-gonal numbers, squares to 4-gonal numbers, and so forth. Using Table 1.2, Nicomachus generalizes one of his previous observations and claims that $p^m{}_n + p^3{}_{n-1} = p^{m+1}{}_n$, where $p^3{}_n$ represents the nth triangular number.

The first translation of the *Introduction* into Latin was done by Apuleius of Madaura shortly after Nicomachus's death, but it did not survive. However, there were a number of commentaries written on the *Introduction*. The most influential, *On Nicomachus's Introduction to Arithmetic*, was written by the fourth century mystic philosopher Iamblichus of Chalcis in Syria. The Islamic world learned of Nicomachus through Thabit ibn Qurra's *Extracts from the Two Books of Nicomachus*. Thabit, a ninth century mathematician, physician, and philosopher, worked at the House of Wisdom in Baghdad and devised an ingenious method to find amicable numbers that we discuss in Chapter 4. A version of the *Introduction* was written by Boethius [beau EE thee us], a Roman philosopher and statesman who was imprisoned by Theodoric King of the Ostrogoths on a charge of conspiracy and put to death in 524. It would be hard to overestimate the influence of Boethius on the cultured and scientific medieval mind. His *De*