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Introduction

1.1. WHAT'S IN THE BOOK?

This book is about using mathematics to think about how humans (and other animals) behave. We are hardly surprised to find that mathematics helps us when we deal with physical things. Although relatively few people can do the relevant mathematics, no one is surprised to find out that buildings are built, airplanes are designed, and ships and cars fueled according to some mathematical principle. But people? Or, for that matter, dogs and birds? Does mathematics have a place in understanding how animate, sentient beings move about, remember, quarrel, live with a spouse, or decide to invest in one venture and not another?

I think it does, and I am going to try to convince you. To make things easier to read, I will use the term *mathematical modeling* to refer to the process of analyzing behavior using the rules of mathematics. Just what this means will be described in more detail later. For now, though, just think of ''mathematical modeling'' as a shorthand for the clumsier term ''using mathematics to study behavior.'' I want to convince you, the reader, that mathematical modeling is often a very good thing to do.

I will proceed by example. The chapters in this book present problems in the social and behavioral sciences, and then show how mathematical modeling has helped us to understand them. Before plunging into the details, though, I want to step back and look at the bigger picture.

Mathematical modeling is a specialization of a bigger idea, using formal analyses to guide actions. This bigger idea has an opponent: the use of memory, pattern recognition, analogies, and informal argument to make a decision. This opponent is no straw person; it's the legendary 800-pound gorilla. Modern psychological research has shown that our brains, and hence our minds, are very well organized to recognize a new situation as ''like what we've seen before,'' and then to use rough-and-ready reasoning to decide what to do. To be fair, we are much better than other animals in our ability to follow abstract, formal

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arguments ... but compared to a computer program for deductive reasoning, we aren't all that good.

The reason computers don't run the world, yet, is that reasoning based on memory and analogy works quickly and often works well. This is especially true when we are dealing with concrete, perceivable situations. Formal analysis shows its strength when we deal with abstractions. It's roughly the difference between deciding which steak to buy at the grocery store and deciding whether or not to invest in beef futures on the Chicago Stock Exchange.

For most of human existence people dealt with beef, not beef futures. Until very recently people could spend their lives moving, lifting, cutting, and building things. While abstract ideas certainly were around, they were not part of very many people's daily affairs. In the Industrial Revolution abstract ideas began to be more important than they had been. The intellectual pace quickened further in the late twentieth century, so much so that the economist Robert Reich has called the modern era the age of the ''symbol analyst.''[1] What he meant by this is that today, an ever-increasing number of people earn their living by manipulating symbols standing for things rather than the things themselves. Issues are decided by analysis rather than memory and pattern recognition. It is becoming more and more important to understand formal analysis, and the ultimate of that analysis, mathematical modeling. A major purpose of this book is to help readers reach such understanding by looking at a variety of models, based on relatively simple mathematics, that have been used to explore social and behavioral issues.

To kick things off, let's take a quick look at some examples showing the advantages and disadvantages of mathematical modeling.

1.2. SOME EXAMPLES OF FORMAL AND INFORMAL THINKING

In the seventeenth century, shipbuilders relied on personal experience to guide ship design. They made drawings of what they wanted without analysis. Skilled laborers then put things together using the drawing as a guide. This method was used in Sweden in 1628 to build the 100-gun *Vasa*, the largest warship of its time. When the King of Sweden saw the plans he had the gut feeling that the ship would be still more powerful if it had an extra gun deck on top. In seventeenth century Sweden, what the king wanted, the king got. The extra gun deck went on forthwith. The *Vasa* sailed the seas, or to be precise, Stockholm harbor, for 30 minutes. Then it capsized. Apparently the king's idea wasn't all that good.

This example does not mean that ''gut'' ideas, based on experience, are always wrong. In classic times, the Romans built their buildings in

[1] Reich (1991).

very much the same way as the *Vasa* was designed. The magnificent Colosseum of Rome, built about A.D. 65, is standing today.

Today every large engineering project relies on mathematical analysis. Indeed, we are not bothered by a decision to let mathematical analysis override our intuitions. On the face of it, the idea that a modern jumbo jet could get off the ground is ridiculous. That was my reaction the first time I saw a Boeing 747, the first of the jumbo jets. Nevertheless, I was not surprised when I read that the 747's test flight went off without a hitch. Why wasn't I surprised? Because I knew that careful mathematical analyses had shown that the 747 would fly. I trusted the mathematics. So do the millions of people who fly every week.

On the other hand, sometimes we are a bit too smug about our abilities to analyze things. This is shown by examples from the ancient and modern art of barrel making.

Back in the seventeenth century, employees of the Prince-Bishop of Würzburg were entitled to a wine ration from His Eminence's cellars. There were complaints that the Prince-Bishop played favorites when he chose the quality of wine to be distributed. He decided to show that these rumors were untrue by constructing a single wine cask, with a diameter of more than 10 meters. Henceforth everyone drew their ration from the same barrel. The barrel still existed in 2000. (I've seen it.) That is impressive, as the Prince-Bishop's barrel makers worked by intuition and custom, just like the designers of the *Vasa*.

Since the nineteenth century, large barrels like this have been built to engineering design, using our knowledge of metal strength, expansion rates, and so forth. And . . .

In the early years of the twentieth century, a massive tank, 15 meters high, was built to store molasses in a factory in Boston. In January 1919 the tank burst. It released a 10-meter wall of molasses, 2 million gallons, on the streets of Boston. Molasses is said to be slow, but if a stream has enough mass behind it, it can push right along. The initial speed of the molasses wave was probably around 50 km/hr. Sadly, 21 people died because they could not outrun it.

The problem was a design fault. The bolts and straps that held the barrel together were made of different metals. On the day of the accident the temperature went from -17° to $+9^\circ$ C (2° to 48° F). Alas, the designers had forgotten to allow for different rates of expansion. The result was the stickiest mess in history. If it had not been for the casualties, this would have been just plain funny.

There are probably thousands of examples where some physical construction or manipulation was made possible by mathematical analysis, and for every thousand of these examples, possibly 10 or 12 where the analysis went wrong. Today the balance is clearly on the side of analysis for physical systems, provided that we use a bit of caution. This is the result of centuries of study in which some of the greatest minds of our species (Euclid, Leonardo, Galileo, Descartes, Newton, and Einstein, to mention a few) developed and applied mathematical analyses to the study of the physical world. Now what about the social world?

1.3. A BIT OF HISTORY

The idea of applying mathematical analyses to the social world is an old one. The very first recorded use of arithmetic, in ancient Assyria, was to solve an economic problem. Assyrian merchants wanted to keep track of goods that were not immediately accessible for inspection. A clay tablet recovered from Assyrian ruins, when translated, said roughly:

> I have paid your agents three minas of silver, so that they may purchase lead for your activities here. Now, if you are still my brother, send me the money owed by courier.[2]

This is clearly mathematics, for it illustrates the use of a medium of exchange, silver, to equate the values of other goods and services. Other tablets from the same era refer to the use of precious metals to value sheep, cattle, and land.

The next example illustrates a more sophisticated use of business mathematics. About 2,000 years later, in the eleventh century, the Spanish hero Ruy Diaz de Bivar (El Cid) needed cash to finance a campaign against the Moors. He sent Martin Antolínez, a nobleman of Burgos, to negotiate a loan from two bankers of that city. Three of the topics for negotiation were, in modern terms, the appropriate surety that El Cid had to put up to secure a loan of 600 marks, the fee that the bankers were to receive for the use of their money, and, interestingly, the finder's fee to be paid to Antolínez. He got 5%, which in modern terms would not be a bad commission. We find the echoes of such activity in modern investment banking and arbitrage.[3]

The Moors against whom El Cid fought were representatives of the sophisticated Arab-Iranian-Mogul civilization that flourished from roughly the eighth until the fifteenth century. Classical Islam's contribution to mathematics was immense. The number system we use today, *Arabic numbers* (which they probably borrowed from India) is well known. Arabic and Iranian scholars also developed the modern concept of algebra. These ideas could be considered contributions to pure mathematics, although clearly much of our applied mathematics would be impossible without them.

[2] Gullberg (1997).
[3] Anonymous ([1100s] 1959), *El Poema de mio Cid*, trans. W. S. Merwin (London: Dent).

Arab scholars of this time also pioneered in introducing mathematical concepts into everyday life. They developed the concept of insurance, which extends investment to the assessment of risk. Insurance is, as we shall see, mathematically virtually identical to gambling, even though it is psychologically quite different. The latter topic caught the fancy of the Europeans. Some of the greatest mathematical minds of the Enlightenment, including Pascal and the Bernoulli brothers, were commissioned to explore strategies for winning gambling games.

In the last two centuries there has been an explosion in the use of modeling to guide our thinking about human affairs. Some of the most interesting cases occur when a mathematical model used to solve a problem in one field is adapted to solve problems in a totally different field. Diagnostic radiologists (physicians who specialize in the interpretation of physiological images, from X rays to magnetic resonance imaging) (MRI) are keenly aware that they can never be certain of a diagnosis, and so must consider both the image they see and the costs of two types of misdiagnoses: false positives (e.g., saying that an organ is cancerous when it is not) and false negatives (failing to spot a tumor). The analytic techniques used to evaluate how well a diagnostic radiologist is doing were developed during World War II as an aid in hunting submarines.

Now, let's take a very different example. In December of 2002 the *New York Times* published an article about the reintroduction of North American wolves into the Yellowstone Park area. According to this article, wildlife biologists believed that in the Yellowstone region a population of 30 breeding pairs of wolves would be sufficient to ensure continuation of the wolf population. Why did they believe that? Because mathematical modeling of wolf population dynamics established that if the number of pairs is greater than 30, the probability that the population level will ever go to zero is acceptably low.

I have been talking about ''mathematical modeling'' without saying exactly what it is. I will now illustrate modeling with a famous physical example, explain it, and look at the general principles it illustrates. In the following chapters the same principles will be applied to problems in economics, ecology, epidemiology, psychology, sociology, and the neurosciences. The topics differ, the models differ, the mathematics differ, but the principles remain the same.

1.4. HOW BIG IS THE EARTH? ERATOSTHENES' SOLUTION

The idea that Columbus showed that the world is round is simply bunk. Columbus conducted a long voyage into an unknown region, and returned. He could have made his voyage on a disk, if the end of the Earth was somewhere to the west of the Americas. The Spanish court never entertained this idea, for neither Columbus nor the Spanish court believed that the world was flat. Three hundred years before Christ was born, the Hellenic Greeks had argued for a spherical Earth, based on (among other things) the observation that ships disappear from sight hull first when they ''sail over the horizon.'' If the ship were sailing on a flat surface its optical image might be diminished as it withdrew, due to limits on sight, and might eventually disappear, but it would do so symmetrically rather than hull first.

The Greeks went well beyond presenting a logical argument for a spherical Earth. Eratosthenes of Cyrene (274–196 B.C.E), the librarian of Alexandria, used a mathematical model to measure the circumference of the Earth. His reasoning and experiments are worth careful study, for they illustrate the principles behind our use of mathematics today. Furthermore, Eratosthenes' principles are as applicable to economic and psychological models today as they were to the geographic model he worked with 2,300 years ago.[4]

To understand what Eratosthenes did, we first have to look at what his predecessor Euclid (330–275 B.C.E.?) had done. Euclid dealt in pure mathematics. He postulated several properties of an abstract world composed of straight lines and points, our modern Euclidean space. Then, in one of the most famous exercises in logic ever written, he used his postulates to prove theorems about the relation of angles, lines, and arcs in that space.

On the basis of Euclid's work, Eratosthenes knew that if you bisect a sphere with a plane, then the cross section of the sphere that cuts the plane is a circle whose center is the center of the sphere. (There is a fancier way to say this; the locus of all points on both the sphere and the plane is a circle.) In the special case of the Earth (Figure 1-1), a subset of these circles consists of (a) all north-south *polar* circumnavigations of the Earth through the poles (i.e., along lines of longitude, switching lines only at a pole) and (b) the equator.

The resulting circle is shown in Figure 1-2, which also shows two points on the circumference of the circle. These correspond either to two points on the equator or two points on the same line of longitude (on the same north-south line from pole to pole.) Therefore, if you want to measure the length of the equator, it is sufficient to measure the length of a polar circumnavigation.

A mathematical model for doing this is shown in Figure 1-2. The figure shows two points, A and S, on the circumference, and a point C at

[4] Many books on the history of mathematics describe Eratosthenes' reasoning. Historians differ as to whether Eratosthenes was brilliant or lucky. His contemporaries seem to have had similarly mixed views of his accomplishments. My account is based largely on the account in Gullberg (1997).

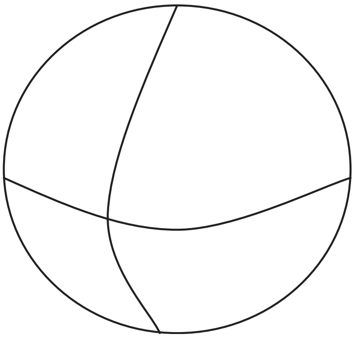

FIGURE 1-1. The first step in Eratosthones' reasoning. If the Earth is a sphere, the equator and any line of longitude can be thought of as points on a circle whose center point is the center of the Earth. All these circles have the same circumference.

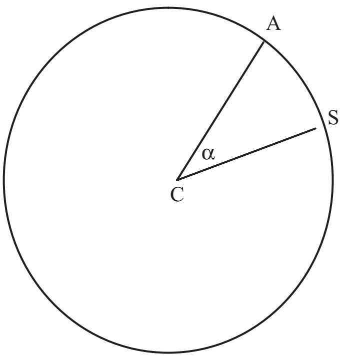

FIGURE 1-2. The second step in Eratosthenes' reasoning. Point C is the center of the circle. If the angle α, at point C, and the length of the arc AS are known, the circumference of the circle can be calculated, using equation (1-1).

the center of the circle, which is also the center of the Earth. Eratosthenes reasoned that the fraction of the circle's circumference that lies on the arc AS is equal to the fraction of the angular measurement of the circle (360° in modern notation) contained in the angle α between lines CA and CS. Translated back into the original problem,

$$\frac{\alpha}{360^\circ} = \frac{arc(AS)}{Cr.}$$
$$Cr = arc(AS) \bullet \frac{360^\circ}{\alpha} \qquad (1\text{-}1)$$

where Cr stands for circumference.

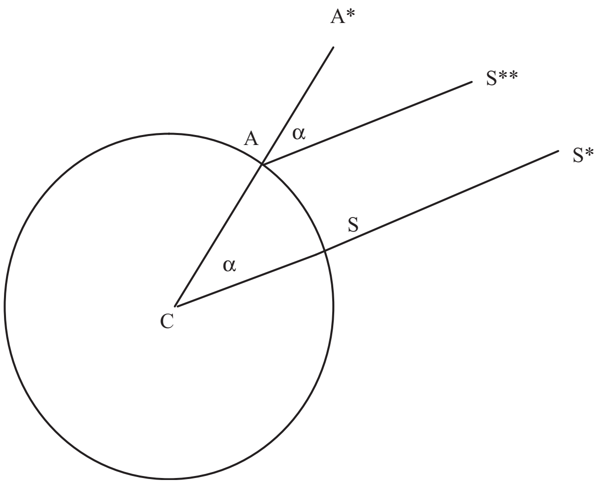

FIGURE 1-3. The third step in Eratosthenes' reasoning. Let lines AS** and SS* be parallel lines (rays of sunlight) and let A and S be two points on a circle with center C. Line SC is an extension of line SS* because the Sun is directly overhead at point S, and so the Sun's rays point directly down toward the center of the Earth. If A* is any point on a line perpendicular to the Earth's surface at point A, then line A*A can also be continued by line AC, which terminates at the center of the Earth. However, line A*A is not parallel to S*A because the Sun is not directly overhead at point A. Therefore, by Euclid's theorem for alternate angles, angle A*AS** = angle ACS = α. Angle A*AS** is on the Earth's surface, and so it can be measured.

Eratosthenes reasoned that if he could find appropriate points A and S on the same line of longitude, and measure angle α, then the length of the Earth's equator could be found. Unfortunately, angle α is at the center of the Earth, ruling out direct measurement. This brings us to the third step in Eratosthenes' reasoning.

The argument is shown in Figure 1-3, which should be examined carefully. Eratosthenes assumed that the Sun's rays are parallel to each other. (This is true if the Earth is much smaller than the Sun, as it is, or if the Sun is very far away, which it is.) Given this assumption, the angle between point A and S, measured at the center of the Earth, C, can be found if we can find two locations on the same longitude (i.e., one directly south of the other). If the Sun is directly overhead at one point, S, (at an angle of incidence of 0°) and the Sun strikes the other point, A, at an angle of incidence of α degrees at exactly the same moment, then the angle between the two, measured from the center of the Earth, is also α. The mathematical argument is shown in Figure 1-3. I urge the reader to examine it carefully.

At this point Eratosthenes had connected his model to reality by expressing it in measurements that could be taken. This can be contrasted with a measure that required, say, suspending an instrument in the sky at a known height, and then measuring the angle between locations using that instrument. We can do this today using satellites and radar-ranging techniques, but the technology was not available to Eratosthenes!

The problem now shifts from one of having a model identifying the measures that need to be made to actually making the measurements. Before Eratosthenes' solution is presented, let us take a look at another problem that he, and every mathematical modeler after him, had to deal with: measurement error.

In order to apply his model, Eratosthenes had to rely on measured values of α and arc(AS) rather than the actual values. Measurements inevitably contain a measurement error. Therefore, any application of equation 1-1 to measured values will be

$$\frac{\alpha + error(\alpha)}{360^\circ} = \frac{arc(AS) + error(arc(AS))}{Cr} \tag{1-2}$$

where *error(α)* and *error(arc(AS))* refer to errors in measuring the angle or the arc. The measurement errors have to be so small relative to their true values that equation (1-2) is close enough to equation (1-1) so that the discrepancy can be disregarded.

There is a general principle here. Application of a model is always limited by our ability to measure the relevant variables! We will meet this idea again, for it certainly is not unique to Eratosthenes' model. The instruments that he had to work with, in the way of measurements of angles and distances, were primitive compared to what we have today. Nevertheless, as will now be shown, he did pretty well considering that neither lasers and radar nor statistics had been invented.

Thought question. Why did I include statistics in that list?

There is another measurement problem, timing. In order for the model to work, the Sun has to be directly overhead at S; that is, the line CS must be a continuation of line SS*. This happens only when the sun is directly overhead at noon. Also, because the Earth and Sun move relative to each other during the day, it is essential that measurements be taken at A, at exactly the time at which the Sun is overhead at S. Unfortunately, a good clock would not be invented until about 1,800 years later (and there wasn't any radio time signal, either), and so Eratosthenes faced another problem. He solved it.

Noon, local time, is the point at which the Sun reaches its maximum height in the sky, at that point. Therefore, if points A and S are on the same longitude, we can make a measure at point A, at local noon, and be sure that the Sun will be at its highest point at S at exactly that time.

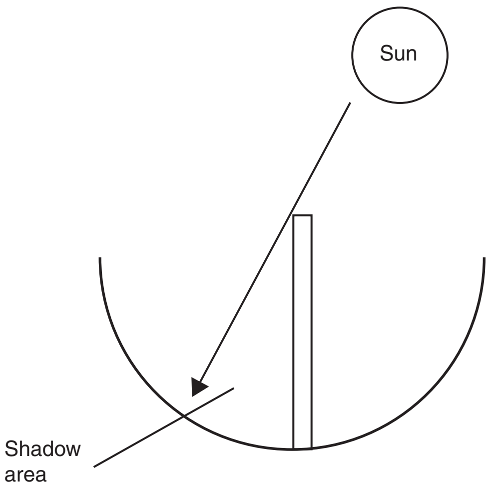

FIGURE 1-4. The straphe was a device used in the Hellenistic period to measure the angle of the Sun. The U-shaped base piece is marked in angular measures. The shaded area indicates how high the Sun is above the horizon. The point at which the Sun is at its height is always local noon. Whether or not the angle is zero, however, depends upon the day and the latitude. For all points on the tropic of Cancer, the Sun is immediately overhead at local noon on the summer solstice.

The device Eratosthenes used to measure angle α was called a *straphe.* It was basically a bowl with a needle sticking up from the middle. The bowl was planted in the ground, with the needle sticking straight up, so that it would be along line CA in Figure 1-3. When this was done, the shadow of the needle would measure angle α. This is shown in Figure 1-4.

Think about it. In Seattle, which is well north of the tropic of Cancer, at noon on the summer solstice (June 21–22) the Sun is always to the south. On the tropic of Cancer, however, the Sun is directly overhead at noon on the summer solstice. Today we know that this is due to the interaction of the Earth's path around the Sun, the angle of inclination of the Earth's axis of rotation to the Sun-Earth line, and the rotation of the Earth. Eratosthenes did not have to know this, although he may well have known of the heliocentric theory developed by Aristarchos of Samos (310–250 B.C.E.) a century earlier. All he needed to do was to know that there was a particular day that marked the solstice, and that this day was the same everywhere.

Eratosthenes next had to find two observation points for A and S. He learned that in the city of Syene (modern Aswan), on the Nile to the south of Alexandria, the Sun shone at the bottom of a vertical well at noon on the summer solstice. This implied that Syene was on the tropic