

THEORY AND MODELING OF ROTATING FLUIDS

A systematic account of the theory and modeling of rotating fluids that highlights the remarkable advances in the area and brings researchers and postgraduate students in atmospheres, oceanography, geophysics, astrophysics and engineering to the frontiers of research. Sufficient mathematical and numerical detail is provided in a variety of geometries, such that the analysis and results can be readily reproduced, and many numerical tables are included to enable readers to compare or benchmark their own calculations. Traditionally, there are two disjointed topics in rotating fluids: convective fluid motion driven by buoyancy, discussed by Chandrasekhar (1961), and inertial waves and precession-driven flow, described by Greenspan (1968). Now, for the first time in book form, the authors present a unified theory for three topics – thermal convection, inertial waves, and precession-driven flow – to demonstrate that these seemingly complicated, and previously disconnected, problems become mathematically simple in the framework of an asymptotic approach that incorporates the essential characteristics of rotating fluids.

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THEORY AND MODELING OF ROTATING FLUIDS

Convection, Inertial Waves and Precession

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Contents

<i>Preface</i>	<i>page xi</i>
Part 1 Fundamentals of Rotating Fluids	1
1 Basic Concepts and Equations for Rotating Fluids	3
1.1 Introduction	3
1.2 Equations of Motion in Rotating Systems	4
1.3 The Heat Equation	6
1.4 The Boussinesq Equations	7
1.5 The Kinetic Energy Equation	10
1.6 Taylor–Proudman Theorem and Thermal Wind Equation	11
1.7 A Unified Approach	13
Part 2 Inertial Waves in Uniformly Rotating Systems	15
2 Introduction	17
2.1 Formulation	17
2.2 Frequency Bound $ \sigma \leq 1$	19
2.3 Special Cases: $\sigma = 0$ and $\sigma = \pm 1$	21
2.4 Orthogonality	23
2.5 The Poincaré Equation	25
3 Inertial Modes in Rotating Narrow-gap Annuli	27
3.1 Formulation	27
3.2 Axisymmetric Inertial Oscillations	29
3.3 Geostrophic Mode	31
3.4 Non-axisymmetric Inertial Waves	32
4 Inertial Modes in Rotating Cylinders	35
4.1 Formulation	35
4.2 Axisymmetric Inertial Oscillations	36
4.3 Geostrophic Mode	41
4.4 Non-axisymmetric Inertial Waves	43
5 Inertial Modes in Rotating Spheres	50
5.1 Formulation	50

vi	<i>Contents</i>	
5.2	Geostrophic Mode	52
5.3	Equatorially Symmetric Modes: $m = 0$	54
5.4	Equatorially Symmetric Modes: $m \geq 1$	60
5.5	Equatorially Antisymmetric Modes: $m = 0$	71
5.6	Equatorially Antisymmetric Modes: $m \geq 1$	75
5.7	An Exact Nonlinear Solution in Rotating Spheres	81
6	Inertial Modes in Rotating Oblate Spheroids	83
6.1	Formulation	83
6.2	Geostrophic Mode	91
6.3	Equatorially Symmetric Modes: $m = 0$	92
6.4	Equatorially Symmetric Modes: $m \geq 1$	94
6.5	Equatorially Antisymmetric Modes: $m = 0$	96
6.6	Equatorially Antisymmetric Modes: $m \geq 1$	99
6.7	An Exact Nonlinear Solution in Rotating Spheroids	102
7	A Proof of Completeness of Inertial Modes in Rotating Channels	105
7.1	Significance of the Completeness of Inertial Modes	105
7.2	Bessel's Inequality and Parseval's Equality	107
7.3	A Proof of the Completeness Relation	109
8	Indications of Completeness of Inertial Modes in Rotating Spheres	118
8.1	Seeking Signs of Completeness	118
8.2	A Proof of the Vanishing Dissipation-type Integral	119
Part 3	Precession and Libration in Non-uniformly Rotating Systems	127
9	Introduction	129
9.1	Non-uniform Rotation: Precession and Libration	129
9.2	Precession/Libration in Different Geometries	130
9.3	Key Parameters and Reference Frames	134
9.4	Asymptotic Expansion Without Using \sqrt{Ek}	135
10	Fluid Motion in Precessing Narrow-gap Annuli	138
10.1	Formulation	138
10.2	Conditions for Resonance	141
10.3	Asymptotic Solution at Resonance with $\Gamma = \sqrt{3}$	142
10.4	Asymptotic Solution at Resonance with $\Gamma = 1/\sqrt{3}$	152
10.5	Linear Numerical Analysis	155
10.6	Nonlinear Direct Numerical Simulation	157
10.7	Comparison: Analytical vs. Numerical	159
10.8	A Byproduct: The Viscous Decay Factor	160
11	Fluid Motion in Precessing Circular Cylinders	164
11.1	Formulation	164
11.2	Conditions for Resonance	166
11.3	Divergence of the Inviscid Precessing Solution	168
11.4	General Asymptotic Solution for $0 < Ek \ll 1$	172

<i>Contents</i>		vii
11.5	Asymptotic Solution at Primary Resonances	180
11.6	Linear Numerical Analysis Using Spectral Methods	187
11.7	Nonlinear Properties of Weakly Precessing Flow	190
11.8	Numerical Simulation Using Finite Element Methods	193
11.9	Nonlinear Precessing Flow at Primary Resonances	195
11.9.1	Decomposition of Nonlinear Flow into Inertial Modes	195
11.9.2	The Structure of Nonlinear Precessing Flow	199
11.9.3	Search for Triadic Resonance	205
11.10	A Byproduct: The Viscous Decay Factor	209
12	Fluid Motion in Precessing Spheres	213
12.1	Formulation	213
12.2	Asymptotic Expansion and Resonance	215
12.3	Asymptotic Solution	217
12.4	Nonlinear Direct Numerical Simulation	224
12.5	Comparison: Analytical vs. Numerical	226
12.6	Nonlinear Effects: Mean Azimuthal Flow	227
12.7	A Byproduct: The Viscous Decay Factor	229
13	Fluid Motion in Longitudinally Librating Spheres	231
13.1	Formulation	231
13.2	Asymptotic Solutions	232
13.2.1	Why Resonance Cannot Occur	232
13.2.2	Asymptotic Analysis	233
13.2.3	Three Fundamental Modes Excited	239
13.3	Linear Numerical Solution	244
13.4	Nonlinear Direct Numerical Simulation	246
14	Fluid Motion in Precessing Oblate Spheroids	250
14.1	Formulation	250
14.2	Inviscid Solution	252
14.3	Exact Nonlinear Solution	258
14.4	Viscous Solution	260
14.5	Properties of Nonlinear Precessing Flow	268
14.6	A Byproduct: The Viscous Decay Factor	273
15	Fluid Motion in Latitudinally Librating Spheroids	276
15.1	Formulation	276
15.2	Analytical Solution: Non-resonant Librating Flow	279
15.3	Analytical Solution: Resonant Librating Flow	283
15.4	Nonlinear Direct Numerical Simulation	293
15.5	Comparison: Analytical vs. Numerical	293
Part 4	Convection in Uniformly Rotating Systems	297
16	Introduction	299
16.1	Rotating Convection vs. Precession/Libration	299

viii	<i>Contents</i>	
16.2	Key Parameters for Rotating Convection	300
16.3	Rotational Constraint on Convection	302
16.4	Types of Rotating Convection	303
16.4.1	Viscous Convection Mode	303
16.4.2	Inertial Convection Mode	305
16.4.3	Transitional Convection Mode	306
16.5	Convection in Various Rotating Geometries	307
16.5.1	Rotating Annular Channels	307
16.5.2	Rotating Circular Cylinders	308
16.5.3	Rotating Spheres or Spherical Shells	309
17	Convection in Rotating Narrow-gap Annuli	313
17.1	Formulation	313
17.2	A Finite-difference Method for Nonlinear Convection	316
17.3	Stationary Viscous Convection	318
17.3.1	Governing Equations	318
17.3.2	Asymptotic Solution for $\Gamma(Ta)^{1/6} \ll O(1)$	320
17.3.3	Asymptotic Solution for $\Gamma(Ta)^{1/6} = O(1)$	325
17.3.4	Numerical Solution Using a Galerkin-tau Method	327
17.3.5	Comparison: Analytical vs. Numerical	329
17.3.6	Nonlinear Properties of Stationary Convection	330
17.4	Oscillatory Viscous Convection	332
17.4.1	Governing Equations	332
17.4.2	Symmetry between Two Different Oscillatory Solutions	334
17.4.3	Asymptotic Solutions Satisfying the Boundary Condition	335
17.4.4	Comparison: Analytical vs. Numerical	343
17.4.5	Comparison with an Unbounded Rotating Layer	348
17.4.6	Nonlinear Properties with $\Gamma = O(Ta^{-1/6})$	352
17.4.7	Nonlinear Properties with $\Gamma \gg O(Ta^{-1/6})$	354
17.5	Viscous Convection with Curvature Effects	356
17.5.1	Onset of Viscous Convection	356
17.5.2	Nonlinear Properties of Viscous Convection	359
17.6	Inertial Convection: Non-axisymmetric Solutions	366
17.6.1	Asymptotic Expansion	366
17.6.2	Non-dissipative Thermal Inertial Wave	367
17.6.3	Asymptotic Solution with Stress-free Condition	369
17.6.4	Asymptotic Solution with No-slip Condition	373
17.6.5	Numerical Solution Using a Galerkin Spectral Method	383
17.6.6	Comparison: Analytical vs. Numerical	385
17.6.7	Nonlinear Properties of Inertial Convection	386
17.7	Inertial Convection: Axisymmetric Torsional Oscillation	393

18	Convection in Rotating Cylinders	396
18.1	Formulation	396
18.2	Convection with Stress-free Condition	399
18.2.1	Asymptotic Solution for Inertial Convection	399
18.2.2	Asymptotic Solution for Viscous Convection	405
18.2.3	Numerical Solution Using a Chebyshev-tau Method	408
18.2.4	Comparison: Analytical vs. Numerical	410
18.3	Convection with No-slip Condition	412
18.3.1	Asymptotic Solution for Inertial Convection	412
18.3.2	Asymptotic Solution for Viscous Convection	418
18.3.3	Numerical Solution Using a Galerkin-type Method	419
18.3.4	Comparison: Analytical vs. Numerical	421
18.3.5	Effect of Thermal Boundary Condition	424
18.3.6	Axisymmetric Inertial Convection	426
18.4	Transition to Weakly Turbulent Convection	430
18.4.1	A Finite Element Method for Nonlinear Convection	430
18.4.2	Inertial Convection: From Single Inertial Mode to Weak Turbulence	431
18.4.3	Viscous Convection: From Sidewall-localized Mode to Weak Turbulence	435
19	Convection in Rotating Spheres or Spherical Shells	439
19.1	Formulation	439
19.2	Numerical Solution using Toroidal/Poloidal Decomposition	442
19.2.1	Governing Equations under Toroidal/Poloidal Decomposition	442
19.2.2	Numerical Analysis for Stress-free or No-slip Condition	444
19.2.3	Several Numerical Solutions for $0 < Ek \ll 1$	447
19.2.4	Nonlinear Effects: Differential Rotation	452
19.3	Local Asymptotic Solution: A Small-gap Annular Model	459
19.3.1	The Local and Quasi-geostrophic Approximation	459
19.3.2	Asymptotic Relation for $0 < Ek \ll 1$	461
19.3.3	Comparison: Asymptotic vs. Numerical	463
19.4	Global Asymptotic Solution with Stress-free Condition	464
19.4.1	Hypotheses for Asymptotic Analysis	464
19.4.2	Asymptotic Analysis for Inertial Convection	465
19.4.3	Several Analytical Solutions for Inertial Convection	470
19.4.4	Differential Rotation Cannot be Sustained by Inertial Convection	474
19.4.5	Asymptotic Analysis for Viscous Convection	476
19.4.6	Typical Asymptotic Solutions for Viscous Convection	479
19.4.7	Nonlinear Effects: Differential Rotation in Viscous Convection	481
19.5	Global Asymptotic Solution with No-slip Condition	484
19.5.1	Hypotheses for Asymptotic Analysis	484
19.5.2	Asymptotic Analysis for Inertial Convection	485

19.5.3	Several Analytical Solutions for Inertial Convection	490
19.5.4	Asymptotic Analysis for Viscous Convection	493
19.5.5	Several Asymptotic Solutions for Viscous Convection	496
19.5.6	Nonlinear Effects: Differential Rotation in Viscous Convection	497
19.6	Transition to Weakly Turbulent Convection	503
19.6.1	A Finite-Element Method for Rotating Spheres	503
19.6.2	Transition to Weak Turbulence in Rotating Spheres	504
19.6.3	A Finite Difference Method for Rotating Spherical Shells	508
19.6.4	Multiple Stable Nonlinear Equilibria in Slowly Rotating Thin Spherical Shells	509
	Appendix A Vector Identities and Theorems	513
	Appendix B Vector Definitions	514
	<i>References</i>	516
	<i>Index</i>	523

Preface

Rotation plays an essential role in the structure and variation of large-scale flows taking place in the interiors, atmospheres and oceans of planets. Knowledge of common hydrodynamical processes in rotating systems constitutes a major necessary component not only in oceanography, but also in planetary and astrophysical sciences. There have been few systematic accounts of the theory of rotating fluids in the more than quarter of a century since Chandrasekhar (1961) and Greenspan (1968) wrote their classic monographs. The second edition of Greenspan's book, Greenspan (1990), was not a major revision. Other volumes, such as the book edited by Roberts and Soward (1978), while containing some interesting articles, did not present the subject in a unified fashion. More recent books by Vanyo (1993) and Boubnov and Golitsyn (1995) mainly concentrate on experimental studies of general rotating flows. Many important developments have taken place in the study of rotating fluids and it has long been necessary to fill a significant gap in the existing literature.

Over the past several decades the subject of rotating fluids has blossomed remarkably and great strides have been made in our understanding of the topic. Not only have there been very many publications on the classic applications of the theory of rotating fluids to the dynamics of atmosphere and oceans, almost exclusively dealing with thin, nearly two dimensional spherical layers of fluids or infinitely unbounded fluid layers, but also considerable attention has been paid to rotating flows in fluid-filled containers such as cylinders, annuli and thick spherical layers or in complete spheres of fluid. Such studies are often considered to be relevant to the dynamics of planetary and stellar interiors and, more importantly, corresponding laboratory experiments in these fluid-filled rotating containers can be carried out, offering deep insight into the understanding of common processes at the heart of rotating flow phenomena.

It is clearly quite impossible today to cover the theory of rotating fluids to the same degree of completeness as Chandrasekhar (1961) or Greenspan (1968) could for the state of the subject in the 1960s. It is necessary to be highly selective. This monograph focuses primarily on the three areas we consider fundamental and central to geophysical and astrophysical applications: (i) inertial waves and oscillations in contained rotating systems caused solely by the effect of rotation, (ii) convective motions controlled by rotation but driven by the thermal instabilities of equilibria in which gravity is parallel to the basic

temperature gradient and (iii) fluid motions forced by non-uniformly rotating systems due to precession or libration. These three topics are physically and mathematically closely related. All the flow phenomena considered in this monograph either take place only in rotating fluids or are critically affected by rotation. It is natural that the selection of topics reflects unavoidably some elements of our personal preferences. We regret that many important subjects, such as compressible flows, the effects of stable stratification and some complex nonlinear theories, are omitted. However, we do believe that the three chosen areas are at the heart of the subject since it is the effect of rotation that controls fluid dynamics in rapidly rotating systems.

Traditionally, there are two disjoint topics in rotating fluids: convective fluid motion driven by buoyancy, discussed by Chandrasekhar (1961), and inertial waves and precession-driven flow, described by Greenspan (1968). This monograph presents a unified theory for three seemingly mathematically and physically disconnected topics – thermal convection, inertial waves, and precession-driven flow – and demonstrates that these apparently complicated problems become mathematically tractable in the framework of an asymptotic approach that incorporates the essential characteristics of rotating fluids.

Our objectives in writing this monograph are threefold. We introduce several topics that have not received extensive attention in other monographs on rotating fluids. We attempt to present a systematic and unified account of the theory of rotating Boussinesq fluids in various geometries as it has developed over the last several decades, hoping that it will lead readers up to the frontiers of research on those topics. Consideration of some geometries such as a rotating annular channel or cylinder is motivated by either their mathematical simplicity and clarity or by possible realization in experimental studies, while studies of other geometries like spheres and spheroids are mainly motivated by their direct application to planetary and astrophysical bodies. Many parts of the monograph are based on our original research over the past decades. Finally, it is also our intention that this monograph should be particularly useful to postgraduate researchers in many different fields such as geophysics, astrophysics, planetary physics and engineering.

With regard to the presentation of the subjects, we follow the following policies on style, depth and details. A considerable effort is made to ensure that each chapter is largely self-contained, readable independently of other chapters, at the expense of often repeating basic equations, certain notations and definitions. In order to make the theoretical development accessible to postgraduate researchers, we have included the amount of mathematical detail that is sufficient for them readily to reproduce the mathematical analysis and results. We also include many numerical tables that enable readers to compare or benchmark their analysis and numerical results if needed. Though we have emphasized the analytical aspects of the theory of rotating fluids, wherever possible we compare the theory with the results of corresponding numerical and experimental studies. We have assumed that readers are familiar with fundamental fluid mechanics, the basic theories of partial and ordinary differential equations and vector analysis.

Much of the material presented in this monograph is based on our own original studies as well as the research of our PhD students and post doctors. KZ is grateful to his

many colleagues, past and present collaborators, throughout the world. In particular, he is indebted to Professors F. H. Busse, D. Gubbins, A. Jackson, C. A. Jones, R. R. Kerswell, P. H. Roberts, G. Schubert and A. M. Soward; through numerous discussions with them over decades, they have significantly influenced his research on the subject. We are also grateful to Dr. S. J. Maskell who has carefully read the manuscript and provided numerous valuable comments. The blame for any defects and errors in this monograph rests squarely on us.

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