Convex Functions: Constructions, Characterizations and Counterexamples

Like differentiability, convexity is a natural and powerful property of functions that plays a significant role in many areas of mathematics, both pure and applied. It ties together notions from topology, algebra, geometry and analysis, and is an important tool in optimization, mathematical programming and game theory. This book, which is the product of a collaboration of over 15 years, is unique in that it focuses on convex functions themselves, rather than on convex analysis. The authors explore the various classes and their characteristics, treating convex functions in both Euclidean and Banach spaces.

They begin by demonstrating, largely by way of examples, the ubiquity of convexity. Chapter 2 then provides an extensive foundation for the study of convex functions in Euclidean (finite-dimensional) space, and Chapter 3 reprises important special structures such as polyhedrality, selection theorems, eigenvalue optimization and semidefinite programming. Chapters 4 and 5 play the same role in (infinite-dimensional) Banach space. Chapter 6 discusses a number of other basic topics, such as selection theorems, set convergence, integral and trace class functionals, and convex functions on Banach lattices.

Chapters 7 and 8 examine Legendre functions and their relation to the geometry of Banach spaces. The final chapter investigates the application of convex functions to (maximal) monotone operators through the use of a recently discovered class of convex representative functions of which the Fitzpatrick function is the progenitor.

The book can either be read sequentially as a graduate text, or dipped into by researchers and practitioners. Each chapter contains a variety of concrete examples and over 600 exercises are included, ranging in difficulty from early graduate to research level.

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To our wives
Judith and Judith

Contents

Preface

This book on *convex functions* emerges out of 15 years of collaboration between the authors. It is far from being the first on the subject nor will it be the last. It is neither a book on *convex analysis* such as Rockafellar's foundational 1970 book [369] nor a book on *convex programming* such as Boyd and Vandenberghe's excellent recent text [128]. There are a number of fine books – both recent and less so – on both those subjects or on *convexity* and relatedly on *variational analysis*. Books such as [371, 255, 378, 256, 121, 96, 323, 332] complement or overlap in various ways with our own focus which is to explore the interplay between the structure of a normed space and the properties of convex functions which can exist thereon. In some ways, among the most similar books to ours are those of Phelps [349] and of Giles [229] in that both also straddle the fields of geometric functional analysis and convex analysis – but without the convex function itself being the central character.

We have structured this book so as to accommodate a variety of readers. This leads to some intentional repetition. Chapter 1 makes the case for the ubiquity of convexity, largely by way of examples, many but not all of which are followed up in later chapters. Chapter 2 then provides a foundation for the study of convex functions in Euclidean (finite-dimensional) space, and Chapter 3 reprises important special structures such as polyhedrality, eigenvalue optimization and semidefinite programming.

Chapters 4 and 5 play the same role in (infinite-dimensional) Banach space. Chapter 6 comprises a number of other basic topics such as Banach space selection theorems, set convergence, integral functionals, trace-class spectral functions and functions on normed lattices.

The remaining three chapters can be read independently of each other. Chapter 7 examines the structure of *Legendre functions* which comprises those barrier functions which are essentially smooth and essentially strictly convex and considers how the existence of such barrier functions is related to the geometry of the underlying Banach space; as always the nicer the space (e.g. is it reflexive, Hilbert or Euclidean?) the more that can be achieved. This coupling between the space and the convex functions which may survive on it is attacked more methodically in Chapter 8.

Chapter 9 investigates (maximal) monotone operators through the use of a specialized class of convex *representative functions* of which the *Fitzpatrick function* is the progenitor. We have written this chapter so as to make it more usable as a stand-alone source on convexity and its applications to monotone operators.

In each chapter we have included a variety of concrete examples and exercises – often guided, some with further notes given in Chapter 10. We both believe strongly that general understanding and intuition rely on having fully digested a good cross-section of particular cases. Exercises that build required theory are often marked with *, those that include broader applications are marked with † and those that take excursions into topics related – but not central to – this book are marked with **.

We think this book can be used as a text, either primary or secondary, for a variety of introductory graduate courses. One possible half-course would comprise Chapters 1, 2, 3 and the finite-dimensional parts of Chapters 4 through 10. These parts are listed at the end of Chapter 3. Another course could encompass Chapters 1 through 6 along with Chapter 8, and so on. We hope also that this book will prove valuable to a larger group of practitioners in mathematical science; and in that spirit we have tried to keep notation so that the infinite-dimensional and finite-dimensional discussion are well comported and so that the book can be dipped into as well as read sequentially. This also requires occasional intentional redundancy. In addition, we finish with a 'bonus chapter' revisiting the boundary between Euclidean and Banach space and making comments on the earlier chapters.

We should like to thank various of our colleagues and students who have provided valuable input and advice and particularly Miroslav Bacak who has assisted us greatly. We should also like to thank Cambridge University Press and especially David Tranah who has played an active and much appreciated role in helping shape this work. Finally, we have a companion web-site at `http://projects.cs.dal.ca/ddrive/ConvexFunctions/` on which various related links and addenda (including any subsequent errata) may be found.