Geometry and Topology

Geometry provides a whole range of views on the universe, serving as the inspiration, technical toolkit and ultimate goal for many branches of mathematics and physics. This book introduces the ideas of geometry, and includes a generous supply of simple explanations and examples. The treatment emphasises coordinate systems and the coordinate changes that generate symmetries. The discussion moves from Euclidean to non-Euclidean geometries, including spherical and hyperbolic geometry, and then on to affine and projective linear geometries. Group theory is introduced to treat geometric symmetries, leading to the unification of geometry and group theory in the Erlangen program. An introduction to basic topology follows, with the Möbius strip, the Klein bottle and the surface with g handles exemplifying quotient topologies and the homeomorphism problem. Topology combines with group theory to yield the geometry of transformation groups, having applications to relativity theory and quantum mechanics. A final chapter features historical discussions and indications for further reading. While the book requires minimal prerequisites, it provides a first glimpse of many research topics in modern algebra, geometry and theoretical physics.

The book is based on many years' teaching experience, and is thoroughly class tested. There are copious illustrations, and each chapter ends with a wide supply of exercises. Further teaching material is available for teachers via the web, including assignable problem sheets with solutions.

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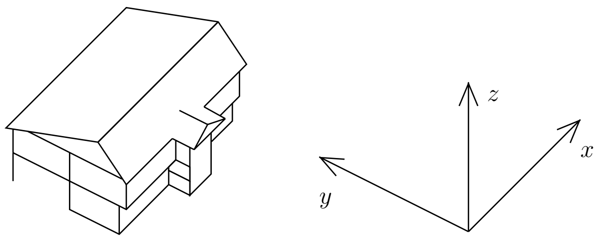
Preface

What is geometry about?

Geometry 'measuring the world' attempts to describe and understand space around us and all that is in it. It is the central activity and main driving force in many branches of math and physics, and offers a whole range of views on the nature and meaning of the universe. This book treats geometry in a wide context, including a wealth of relations with surrounding areas of math and other aspects of human experience.

Any discussion of geometry involves tension between the twin ideals of intuition and precision. *Descriptive* or *synthetic* geometry takes as its starting point our ideas and experience of the observed world, and treats geometric objects such as lines and shapes as objects in their own right. For example, a line could be the path of a light ray in space; you can envisage comparing line segments or angles by 'moving' one over another, thus giving rise to notions of 'congruent' figures, equal lengths, or equal angles that are independent of any quantitative measurement. If A, B, C are points along a line segment, what it means for B to be *between* A and C is an idea hard-wired into our consciousness. While descriptive geometry is intuitive and natural, and can be made mathematically rigorous (and, of course, Euclidean geometry was studied in these terms for more than two millennia, compare 9.1), this is not my main approach in this book.

My treatment centres rather on *coordinate* geometry. This uses Descartes' idea (1637) of measuring distances to view points of space and geometric quantities in terms of numbers, with respect to a fixed origin, using intuitive ideas such as 'a bit to the right' or 'a long way up' and using them quantitatively in a systematic and precise way. In other words, I set up the (x, y)-plane $\mathbb{R}^2$, the (x, y, z)-space $\mathbb{R}^3$ or whatever I need, and use it as a mathematical model of the plane (space, etc.), for the purposes of calculations. For example, to plan the layout of a car park, I might map it onto a sheet of paper or a computer screen, pretending that pairs (x, y) of real numbers correspond to points of the surface of the earth, at least in the limited region for which I have planning permission. Geometric constructions, such as drawing an even rectangular grid or planning the position of the ticket machines to ensure the maximum aggravation to customers, are easier to make in the model than in real



A coordinate model of space.

life. We admit possible drawbacks of our model, but its use divides any problem into calculations within the model, and considerations of how well it reflects the practical world.

Topology is the youngster of the geometry family. Compared to its venerable predecessors, it really only got going in the twentieth century. It dispenses with practically all the familiar quantities central to other branches of geometry, such as distance, angles, cross-ratios, and so on. If you are tempted to the conclusion that there is not much left for topology to study, think again. Whether two loops of string are linked or not does not depend on length or shape or perspective; if that seems too simple to be a serious object of study, what about the linking or knotting of strands of DNA, or planning the over- and undercrossings on a microchip? The higher dimensional analogues of disconnecting or knotting are highly nontrivial and not at all intuitive to denizens of the lower dimensions such as ourselves, and cannot be discussed without formal apparatus. My treatment of topology runs briefly through abstract *point-set topology*, a fairly harmless generalisation of the notion of continuity from a first course on analysis and metric spaces. However, my main interest is in topology as *rubber-sheet geometry*, dealing with manifestly geometric ideas such as closed curves, spheres, the torus, the Möbius strip and the Klein bottle.

Change of coordinates, motions, group theory and the Erlangen program

Descartes' idea to use numbers to describe points in space involves the choice of a *coordinate system* or *coordinate frame*: an origin, together with axes and units of length along the axes. A recurring theme of all the different geometries in this book is the question of what a coordinate frame is, and what I can get out of it. While coordinates provide a convenient framework to discuss points, lines, and so on, it is a basic requirement that any meaningful statement in geometry is *independent of the choice of coordinates*. That is, coordinate frames are a humble technical aid in determining the truth, and are not allowed the dignity of having their own meaning.

Changing from one coordinate frame to another can be viewed as a *transformation* or *motion*: I can use a motion of space to align the origin and coordinate axes of two coordinate systems. A statement that remains true under any such motion is independent of the choice of coordinates. Felix Klein's 1872 *Erlangen program* formalises this relation between geometric properties and changes of coordinates by defining geometry to be the study of properties invariant under allowed coordinate changes, that is, invariant under a *group of transformations*. This approach is closely related to the point of view of special relativity in theoretical physics (Einstein, 1905), which insists that the laws of physics must be invariant under Lorentz transformations.

This course discusses several different geometries: in some case the spaces themselves are different (for example, the sphere and the plane), but in others the difference is purely in the conventions I make about coordinate changes. Metric geometries such as Euclidean and hyperbolic non-Euclidean geometry include the notions of distance between two points and angle between two lines. The allowed transformations are rigid motions (isometries or congruences) of Euclidean or hyperbolic space. Affine and projective geometries consider properties such as collinearity of points, and the typical group is the general linear group $\mathrm{GL}(n)$, the group of invertible $n \times n$ matrixes. Projective geometry presents an interesting paradox: while its mathematical treatment involves what may seem to be quite arcane calculations, your brain has a sight driver that carries out projective transformations by the thousand every time you recognise an object in perspective, and does so unconsciously and practically instantaneously.

The sets of transformations that appear in topology, for example the set of all continuous one-to-one maps of the interval [0, 1] to itself, or the same thing for the circle S^1 or the sphere S^2, are of course too big for us to study by analogy with transformation groups such as $\mathrm{GL}(n)$ or the Euclidean group, whose elements depend on finitely many parameters. In the spirit of the Erlangen program, properties of spaces that remain invariant under such a huge set of equivalences must be correspondingly coarse. I treat a few basic topological properties such as compactness, connectedness, winding number and simple connectedness that appear in many different areas of analysis and geometry. I use these simple ideas to motivate the central problem of topology: how to distinguish between topologically different spaces? At a more advanced level, topology has developed systematic invariants that apply to this problem, notably the fundamental group and homology groups. These are invariants of spaces that are the same for topologically equivalent spaces. Thus if you can calculate one of these invariants for two spaces (for example, a disc and a punctured disc) and prove that the answers are different, then the spaces are certainly not topologically equivalent. You may want to take subsequent courses in topology to become a real expert, and this course should serve as a useful guide in this.

Geometry in applications

Although this book is primarily intended for use in a math course, and the topics are oriented towards the theoretical foundations of geometry, I must stress that the math ideas discussed here are applicable in different ways, basic or sophisticated, as stated or with extra development, on their own or in combination with other disciplines, Euclidean or non-Euclidean, metric or topological, to a huge variety of scientific and technological problems in the modern world. I discuss in Chapter 8 the quantum mechanical description of the electron that illustrates a fundamental application of the ideas of group theory and topology to the physics of elementary particles. To move away from basic to more applied science, let me mention a few examples from technology. The typesetting and page layout software now used throughout the newspaper and publishing industry, as well as in the computer rooms of most university departments, can obviously not exist without a knowledge of basic coordinate geometry: even a primary instruction such as 'place letter A or box B, scaled by such-and-such a factor, slanted at such-and-such an angle, at such-and-such a point on the page' involves affine transformations. Within the same industry, computer typefaces themselves are designed using Bezier curves. The geometry used in robotics is more sophisticated. The technological aim is, say, to get a robot arm holding a spanner into the right position and orientation, by adjusting some parameters, say, angles at joints or lengths of rods. This translates in a fairly obvious way into the geometric problem of parametrising a piece of the Euclidean group; but the solution or approximate solution of this problem is hard, involving the topology and analysis of manifolds, algebraic geometry and singularity theory. The computer processing of camera images, whose applications include missile guidance systems, depends among other things on projective transformations (I say this for the benefit of students looking for a career truly worthy of their talents and education). Although scarcely having the same nobility of purpose, similar techniques apply in ultrasonic scanning used in antenatal clinics; here the geometric problem is to map the variations in density in a 3-dimensional medium onto a 2-dimensional computer screen using ultrasonic radar, from which the human eye can easily make out salient features. By a curious coincidence, 3 hours before I, the senior author, gave the first lecture of this course in January 1989, I was at the maternity clinic of Walsgrave hospital Coventry looking at just such an image of a 16-week old foetus, now my third daughter Murasaki.

About this book

Who the book is for

This book is intended for the early years of study of an undergraduate math course. For the most part, it is based on a second year module taught at Warwick over many years, a module that is also taken by first and third year math students, and by students from the math/physics course. You will find the book accessible if you are familiar with most of the following, which is standard material in first and second year math courses.

Coordinate geometry How to express lines and circles in $\mathbb{R}^2$ in terms of coordinates, and calculate their points of intersection; some idea of how to do the same in $\mathbb{R}^3$ and maybe $\mathbb{R}^n$ may also be helpful.

Linear algebra Vector spaces and linear maps over $\mathbb{R}$ and $\mathbb{C}$, bases and matrixes, change of bases, eigenvalues and eigenvectors. This is the only major piece of math that I take for granted. The examples and exercises make occasional reference to vector spaces over fields other than $\mathbb{R}$ or $\mathbb{C}$ (such as finite fields), but you can always omit these bits if they make you uncomfortable.

Multilinear algebra Bilinear and quadratic forms, and how to express them in matrix terms; also Hermitian forms. I summarise all the necessary background material in Appendix B.

Metric spaces Some prior familiarity with the first ideas of a metric space course would not do any harm, but this is elementary material, and Appendix A contains all that you need to know.

Group theory I have gone to some trouble to develop from first principles all the group theory that I need, with the intention that my book can serve as a first introduction to transformation groups and the notions of abstract group theory if you have never seen these. However, if you already have some idea of basic things such as composition laws, subgroups, cosets and the symmetric group, these will come in handy as motivation. If you prefer to see a conventional introduction to group theory, there are any number of textbooks, for example Green [10] or Ledermann [14]. If you intend to study group theory beyond the introductory stage, I strongly recommend Artin [1] or Segal [22]. My ideological slant on this issue is discussed in more detail in 9.2.

How to use the book Although the thousands queueing impatiently at supermarkets and airport bookshops to get their hands on a copy of this book for vacation reading was strong motivation for me in writing it, experience suggests the harsher view of reality: at least some of my readers may benefit from coercion in the form of an organised lecture course.

Experience from teaching at Warwick shows that Chapters 1–6 make a reasonably paced 30 hour second year lecture course. Some more meat could be added to subjects that the lecturer or students find interesting; reflection groups following Coxeter [5], Chapter 4 would be one good candidate. Topics from Chapters 7–8 or the further topics of Chapter 9 could then profitably be assigned to students as essay or project material. An alternative course oriented towards group theory could start with affine and Euclidean geometry and some elements of topology (maybe as a refresher), and concentrate on Chapters 3, 6 and 8, possibly concluding with some material from Segal [22]. This would provide motivation and techniques to study matrix groups from a geometric point of view, one often ignored in current texts.

The author's identity crisis I want the book to be as informal as possible in style. To this end, I always refer to the student as 'you', which has the additional advantage that it is independent of your gender and number. I also refer to myself by the first person singular, despite the fact that there are two of me. Each of me has lectured the material many times, and is used to taking personal responsibility for the truth of my assertions. My model is van der Waerden's style, who always wrote the crisp 'Ich behaupte . . . ' (often when describing results he learned from Emmy Noether or Emil Artin's lectures). I leave you to imagine the speaker as your ideal teacher, be it a bearded patriarch or a fresh-faced bespectacled Central European intellectual.

Acknowledgements A second year course with the title 'Geometry' or 'Geometry and topology' has been given at Warwick since the 1960s. It goes without saying that my choice of material, and sometimes the material itself, is taken in part from the experience of colleagues, including John Jones, Colin Rourke, Brian Sanderson; David Epstein has also provided some valuable material, notably in the chapter on hyperbolic geometry. I have also copied material consciously or unconsciously from several of the textbooks recommended for the course, in particular Coxeter [5], Rees [19], Nikulin and Shafarevich [18] and Feynman [7]. I owe special thanks to Katrin Wendland, the most recent lecturer of the Warwick course MA243 Geometry, who has provided a detailed criticism of my text, thereby saving me from a variety of embarrassments.

Disclaimer Wen solche Lehren nicht erfreun,
Verdienet nicht ein Mensch zu sein.

From Sarastro's aria, The Magic Flute, II.3.

This is an optional course. If you don't like my teaching, please deregister before the deadline.