

AN INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

A complete introduction to partial differential equations, this textbook provides a rigorous yet accessible guide to students in mathematics, physics and engineering. The presentation is lively and up to date, with particular emphasis on developing an appreciation of underlying mathematical theory.

Beginning with basic definitions, properties and derivations of some fundamental equations of mathematical physics from basic principles, the book studies first-order equations, the classification of second-order equations, and the one-dimensional wave equation. Two chapters are devoted to the separation of variables, whilst others concentrate on a wide range of topics including elliptic theory, Green's functions, variational and numerical methods.

A rich collection of worked examples and exercises accompany the text, along with a large number of illustrations and graphs to provide insight into the numerical examples.

Solutions and hints to selected exercises are included for students whilst extended solution sets are available to lecturers from solutions@cambridge.org.

Cambridge University Press
0521848865 - An Introduction to Partial Differential Equations
Yehuda Pinchover and Jacob Rubinstein
Frontmatter
[More information](#)

AN INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

YEHUDA PINCHOVER AND JACOB RUBINSTEIN



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press
0521848865 - An Introduction to Partial Differential Equations
Yehuda Pinchover and Jacob Rubinstein
Frontmatter
[More information](#)

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK

www.cambridge.org
information on this title: www.cambridge.org/9780521848865

© Cambridge University Press 2005

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 2005

Printed in the United Kingdom at the University Press, Cambridge

A catalog record for this book is available from the British Library

Library of Congress Cataloging in Publication data

ISBN-13 978-0-521-84886-2 hardback
ISBN-10 0-521-84886-5 hardback

ISBN-13 978-0-521-61323-X paperback
ISBN-10 0-521-61323-1 paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or
third-party internet websites referred to in this book, and does not guarantee that any content on such websites is,
or will remain, accurate or appropriate.

Cambridge University Press
0521848865 - An Introduction to Partial Differential Equations
Yehuda Pinchover and Jacob Rubinstein
Frontmatter
[More information](#)

To our parents

*The equation of heaven and earth
remains unsolved.*
(Yehuda Amichai)

המשוואה של שמים וארץ
נשארת דלי פתרון.
(יהודה עמיחי)

Contents

	<i>Preface</i>	<i>page xi</i>
1	Introduction	1
	1.1 Preliminaries	1
	1.2 Classification	3
	1.3 Differential operators and the superposition principle	3
	1.4 Differential equations as mathematical models	4
	1.5 Associated conditions	17
	1.6 Simple examples	20
	1.7 Exercises	21
2	First-order equations	23
	2.1 Introduction	23
	2.2 Quasilinear equations	24
	2.3 The method of characteristics	25
	2.4 Examples of the characteristics method	30
	2.5 The existence and uniqueness theorem	36
	2.6 The Lagrange method	39
	2.7 Conservation laws and shock waves	41
	2.8 The eikonal equation	50
	2.9 General nonlinear equations	52
	2.10 Exercises	58
3	Second-order linear equations in two independent variables	64
	3.1 Introduction	64
	3.2 Classification	64
	3.3 Canonical form of hyperbolic equations	67
	3.4 Canonical form of parabolic equations	69
	3.5 Canonical form of elliptic equations	70
	3.6 Exercises	73

viii	<i>Contents</i>	
4	The one-dimensional wave equation	76
4.1	Introduction	76
4.2	Canonical form and general solution	76
4.3	The Cauchy problem and d'Alembert's formula	78
4.4	Domain of dependence and region of influence	82
4.5	The Cauchy problem for the nonhomogeneous wave equation	87
4.6	Exercises	93
5	The method of separation of variables	98
5.1	Introduction	98
5.2	Heat equation: homogeneous boundary condition	99
5.3	Separation of variables for the wave equation	109
5.4	Separation of variables for nonhomogeneous equations	114
5.5	The energy method and uniqueness	116
5.6	Further applications of the heat equation	119
5.7	Exercises	124
6	Sturm–Liouville problems and eigenfunction expansions	130
6.1	Introduction	130
6.2	The Sturm–Liouville problem	133
6.3	Inner product spaces and orthonormal systems	136
6.4	The basic properties of Sturm–Liouville eigenfunctions and eigenvalues	141
6.5	Nonhomogeneous equations	159
6.6	Nonhomogeneous boundary conditions	164
6.7	Exercises	168
7	Elliptic equations	173
7.1	Introduction	173
7.2	Basic properties of elliptic problems	173
7.3	The maximum principle	178
7.4	Applications of the maximum principle	181
7.5	Green's identities	182
7.6	The maximum principle for the heat equation	184
7.7	Separation of variables for elliptic problems	187
7.8	Poisson's formula	201
7.9	Exercises	204
8	Green's functions and integral representations	208
8.1	Introduction	208
8.2	Green's function for Dirichlet problem in the plane	209
8.3	Neumann's function in the plane	219
8.4	The heat kernel	221
8.5	Exercises	223

Contents

ix

9	Equations in high dimensions	226
9.1	Introduction	226
9.2	First-order equations	226
9.3	Classification of second-order equations	228
9.4	The wave equation in \mathbb{R}^2 and \mathbb{R}^3	234
9.5	The eigenvalue problem for the Laplace equation	242
9.6	Separation of variables for the heat equation	258
9.7	Separation of variables for the wave equation	259
9.8	Separation of variables for the Laplace equation	261
9.9	Schrödinger equation for the hydrogen atom	263
9.10	Musical instruments	266
9.11	Green's functions in higher dimensions	269
9.12	Heat kernel in higher dimensions	275
9.13	Exercises	279
10	Variational methods	282
10.1	Calculus of variations	282
10.2	Function spaces and weak formulation	296
10.3	Exercises	306
11	Numerical methods	309
11.1	Introduction	309
11.2	Finite differences	311
11.3	The heat equation: explicit and implicit schemes, stability, consistency and convergence	312
11.4	Laplace equation	318
11.5	The wave equation	322
11.6	Numerical solutions of large linear algebraic systems	324
11.7	The finite elements method	329
11.8	Exercises	334
12	Solutions of odd-numbered problems	337
A.1	Trigonometric formulas	361
A.2	Integration formulas	362
A.3	Elementary ODEs	362
A.4	Differential operators in polar coordinates	363
A.5	Differential operators in spherical coordinates	363
	<i>References</i>	364
	<i>Index</i>	366

Preface

This book presents an introduction to the theory and applications of partial differential equations (PDEs). The book is suitable for all types of basic courses on PDEs, including courses for undergraduate engineering, sciences and mathematics students, and for first-year graduate courses as well.

Having taught courses on PDEs for many years to varied groups of students from engineering, science and mathematics departments, we felt the need for a textbook that is concise, clear, motivated by real examples and mathematically rigorous. We therefore wrote a book that covers the foundations of the theory of PDEs. This theory has been developed over the last 250 years to solve the most fundamental problems in engineering, physics and other sciences. Therefore we think that one should not treat PDEs as an abstract mathematical discipline; rather it is a field that is closely related to real-world problems. For this reason we strongly emphasize throughout the book the relevance of every bit of theory and every practical tool to some specific application. At the same time, we think that the modern engineer or scientist should understand the basics of PDE theory when attempting to solve specific problems that arise in applications. Therefore we took great care to create a balanced exposition of the theoretical and applied facets of PDEs.

The book is flexible enough to serve as a textbook or a self-study book for a large class of readers. The first seven chapters include the core of a typical one-semester course. In fact, they also include advanced material that can be used in a graduate course. Chapters 9 and 11 include additional material that together with the first seven chapters fits into a typical curriculum of a two-semester course. In addition, Chapters 8 and 10 contain advanced material on Green's functions and the calculus of variations. The book covers all the classical subjects, such as the separation of variables technique and Fourier's method (Chapters 5, 6, 7, and 9), the method of characteristics (Chapters 2 and 9), and Green's function methods (Chapter 8). At the same time we introduce the basic theorems that guarantee that the problem at

hand is well defined (Chapters 2–10), and we took care to include modern ideas such as variational methods (Chapter 10) and numerical methods (Chapter 11).

The first eight chapters mainly discuss PDEs in two independent variables. Chapter 9 shows how the methods of the first eight chapters are extended and enhanced to handle PDEs in higher dimensions. Generalized and weak solutions are presented in many parts of the book.

Throughout the book we illustrate the mathematical ideas and techniques by applying them to a large variety of practical problems, including heat conduction, wave propagation, acoustics, optics, solid and fluid mechanics, quantum mechanics, communication, image processing, musical instruments, and traffic flow.

We believe that the best way to grasp a new theory is by considering examples and solving problems. Therefore the book contains hundreds of examples and problems, most of them at least partially solved. Extended solutions to the problems are available for course instructors using the book from solutions@cambridge.org. We also include dozens of drawing and graphs to explain the text better and to demonstrate visually some of the special features of certain solutions.

It is assumed that the reader is familiar with the calculus of functions in several variables, with linear algebra and with the basics of ordinary differential equations. The book is almost entirely self-contained, and in the very few places where we cannot go into details, a reference is provided.

The book is the culmination of a slow evolutionary process. We wrote it during several years, and kept changing and adding material in light of our experience in the classroom. The current text is an expanded version of a book in Hebrew that the authors published in 2001, which has been used successfully at Israeli universities and colleges since then.

Our cumulative expertise of over 30 years of teaching PDEs at several universities, including Stanford University, UCLA, Indiana University and the Technion – Israel Institute of Technology guided us to create a text that enhances not just technical competence but also deep understanding of PDEs. We are grateful to our many students at these universities with whom we had the pleasure of studying this fascinating subject. We hope that the readers will also learn to enjoy it.

We gratefully acknowledge the help we received from a number of individuals. Kristian Jenssen from North Carolina State University, Lydia Peres and Tiferet Saadon from the Technion – Israel Institute of Technology, and Peter Sternberg from Indiana University read portions of the draft and made numerous comments and suggestions for improvement. Raya Rubinstein prepared the drawings, while Yishai Pinchover and Aviad Rubinstein assisted with the graphs. Despite our best efforts, we surely did not discover all the mistakes in the draft. Therefore we encourage observant readers to send us their comments at pincho@techunix.technion.ac.il. We will maintain a webpage with a list of errata at <http://www.math.technion.ac.il/~pincho/PDE.pdf>.