Boolean Functions

Written by prominent experts in the field, this monograph provides the first comprehensive and unified presentation of the structural, algorithmic, and applied aspects of the theory of Boolean functions.

The book focuses on algebraic representations of Boolean functions, especially disjunctive and conjunctive normal form representations. It presents within this framework the fundamental elements of the theory (Boolean equations and satisfiability problems, prime implicants and associated short representations, dualization), an in-depth study of special classes of Boolean functions (quadratic, Horn, shellable, regular, threshold, read-once functions and their characterization by functional equations), and two fruitful generalizations of the concept of Boolean functions (partially defined functions and pseudo-Boolean functions). Several topics are presented here in book form for the first time.

Because of the unique depth and breadth of the unified treatment that it provides and its emphasis on algorithms and applications, this monograph will have special appeal for researchers and graduate students in discrete mathematics, operations research, computer science, engineering, and economics.

Dr. Yves Crama is Professor of Operations Research and Production Management and the former Director General of the HEC Management School of the University of Liège, Belgium. He is widely recognized as a prominent expert in the field of Boolean functions, combinatorial optimization, and operations research, and he has coauthored more than seventy papers and three books on these subjects. Dr. Crama is a member of the editorial board of *Discrete Applied Mathematics, Discrete Optimization, Journal of Scheduling, and 4OR – The Quarterly Journal of the Belgian, French and Italian Operations Research Societies*.

The late Peter L. Hammer (1936–2006) was a Professor of Operations Research, Mathematics, Computer Science, Management Science, and Information Systems at Rutgers University and the Director of the Rutgers University Center for Operations Research (RUTCOR). He was the founder and editor-in-chief of the journals *Annals of Operations Research, Discrete Mathematics, Discrete Applied Mathematics, Discrete Optimization,* and *Electronic Notes in Discrete Mathematics*. Dr. Hammer was the initiator of numerous pioneering investigations of the use of Boolean functions in operations research and related areas, of the theory of pseudo-Boolean functions, and of the logical analysis of data. He published more than 240 papers and 19 books on these topics.

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Boolean Functions

Theory, Algorithms, and Applications

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To Edith,
by way of apology for countless days
spent in front of the computer.
YC

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Preface

Boolean functions, meaning $\{0,1\}$-valued functions of a finite number of $\{0,1\}$-valued variables, are among the most fundamental objects investigated in pure and applied mathematics. Their importance can be explained by several interacting factors.

- It is reasonable to argue that a multivariate function $f : A_1 \times A_2 \times \ldots \times A_n \rightarrow A$ is "interesting" only if each of the sets $A_1, A_2, \ldots, A_n$, and A contains at least two elements, since otherwise the function either depends trivially on some of its arguments, or is constant. Thus, in a sense, Boolean functions are the "simplest interesting" multivariate functions. It may even be surprising, actually, that such primitive constructs turn out to display a rich array of properties and have been investigated by various breeds of scientists for more than 150 years.
- When the arguments of a Boolean function are viewed as atomic logical propositions, the value of the function at a 0–1 point can be interpreted as the truth value of a sentence composed from these propositions. Carrying out calculations on Boolean functions is then tantamount to performing related logical operations (such as inference or theorem-proving) on propositional sentences. Therefore, Boolean functions are at the heart of propositional logic.
- Many concepts of combinatorial analysis have their natural Boolean counterpart. In particular, since every 0–1 point with n coordinates can be viewed as the characteristic vector of a subset of $N = \{1,2,\ldots,n\}$, the set of points at which a Boolean function takes value 1 corresponds to a collection of subsets of N, or a "hypergraph" on N. (When all subsets have cardinality 2, then the function corresponds exactly to a graph.) Structural properties relating to the transversals, stable sets, or colorings of the hypergraph, for instance, often translate into interesting properties of the Boolean function.
- Boolean functions are ubiquitous in theoretical computer science, where they provide fundamental models for the most basic operations performed by

computers on binary digits (or bits). Turing machines and Boolean circuits are prime examples illustrating this claim. Similarly, electrical engineers rely on the Boolean formalism for the description, synthesis, or verification of digital circuits.

- In operations research or management science, binary variables and Boolean functions are frequently used to formulate problems where a number of "go – no go" decisions are to be made; these could be, for instance, investment decisions arising in a financial management framework, or location decisions in logistics, or assignment decisions for production planning. In most cases, the variables have to be fixed at values that satisfy constraints expressible as Boolean conditions and that optimize an appropriate real-valued objective function. This leads to – frequently difficult – Boolean equations ("satisfiability problems") or integer programming problems.
- Voting games and related systems of collective choice are frequently represented by Boolean functions, where the variables are associated with (binary) alternatives available to the decision makers, and the value of the function indicates the outcome of the process.
- Various branches of artificial intelligence rely on Boolean functions to express deductive reasoning processes (in the above-mentioned propositional framework), or to model primitive cognitive and memorizing activities of the brain by neural networks, or to investigate efficient learning strategies, or to devise storing and retrieving mechanisms in databases, and so on.

We could easily extend this list to speak of Boolean models arising in reliability theory, in cryptography, in coding theory, in multicriteria analysis, in mathematical biology, in image processing, in theoretical physics, in statistics, and so on.

The main objective of the present monograph is to introduce the reader to the fundamental elements of the theory of Boolean functions. It focuses on algebraic representations of Boolean functions, especially disjunctive or conjunctive normal form expressions, and it provides a very comprehensive presentation of the structural, algorithmic, and applied aspects of the theory in this framework.

The monograph is divided into three main parts.

Part I: *Foundations* proposes in Chapter 1: *Fundamental concepts and applications,* an introduction to the major concepts and applications of the theory. It then successively tackles three generic classes of problems that play a central role in the theory and in the applications of Boolean functions, namely, Boolean equations and their extensions in Chapter 2: *Boolean equations*, the generation of prime implicants and of optimal normal form representations in Chapter 3: *Prime implicants and minimal DNFs*, and various aspects of the relation between functions and their dual in Chapter 4: *Duality theory*.

Part II: *Special Classes* presents an in-depth study of several remarkable classes of Boolean functions. Each such class is investigated from both the structural and the algorithmic points of view. Chapter 5 is devoted to *Quadratic functions*, Chapter 6 to *Horn functions*, Chapter 7 to *Orthogonal forms and shellability*, Chapter 8 to

Regular functions, Chapter 9 to *Threshold functions*, and Chapter 10 to *Read-once functions*. Chapter 11: *Characterizations of special classes by functional equations* provides general conditions under which classes of functions can be "compactly" characterized.

Finally, Part III: *Generalizations* deals with two fruitful extensions of the concept of Boolean functions. Namely, Chapter 12: *Partially defined Boolean functions* deals with functions whose domain is restricted to a subset of all possible $\{0,1\}$ points, and Chapter 13: *Pseudo-Boolean functions* proposes a brief overview of the theory of real-valued functions of binary variables.

In view of its emphasis on algorithms and applications, this monograph should appeal to researchers and graduate students in discrete mathematics, operations research, computer science, engineering, and economics. Although we believe that it is rather unique in its depth and breadth, our work has been influenced in various ways by many other books dealing with specialized aspects of the field, such as threshold logic, logical inference, operations research, game theory, or reliability theory. We like to mention, in particular, the classic monograph by P.L. Hammer and S. Rudeanu, *Boolean Methods in Operations Research and Related Areas* (Springer, Berlin, 1968). Although it focuses almost exclusively on Boolean models, rather than pseudo-Boolean ones, it can be seen as a distant follow-up to the 1968 monograph. We should also cite the influence of books by Anthony [25]; Brayton, Hachtel, McMullen, and Sangiovanni-Vincentelli [153]; Brown [156]; Chandru and Hooker [184]; Chang and Lee [186]; Hu [511, 512]; Jeroslow [533]; Kleine, Büning, and Lettmann [571]; Knuth [575]; Mendelson [680]; Muroga [698, 699]; Ramamurthy [777]; Rudeanu [795, 796]; Schneeweiss [811]; Störmer [849]; Truemper [871]; Wegener [902, 903]; and Winder [917], among others.

As a complement to the monograph, the reader is also advised to consult the collection of papers *Boolean Models and Methods in Mathematics, Computer Science and Engineering* (Y. Crama and P.L. Hammer, eds., Cambridge University Press, Cambridge, UK, 2010). Each chapter in that volume introduces the reader to specialized Boolean models and applications investigated in a particular field of science and provides a survey of important representative results.

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Comments, reviews, and corrections have been provided at different stages by colleagues and by RUTCOR students, including Nina Feferman, Noam Goldberg, Levent Kandiller, Shaoji Li, Tongyin Liu, Irina Lozina, Martin Milanic, Devon Morrese, David Neu, Sergiu Rudeanu, Gábor Rudolf, Jan-Georg Smaus, and Mine Subasi.

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Finally, I am deeply indebted to my mentor, colleague, and friend, Peter L. Hammer, for getting us started on this ambitious project, many years ago. Peter spent much of his academic career stressing the importance and relevance of Boolean models in different fields of applied mathematics, and he was very keen on completing this monograph. It is extremely unfair that he did not live to see the outcome of our joint effort. I am sure that he would have loved it, and that he would have been very proud of this contribution to the dissemination of the theory, algorithms, and applications of Boolean functions.

Yves Crama
Liège, Belgium, September 2010

Notations

$\mathcal{B} = \{0,1\}$, $U = [0,1]$

$X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n), \ldots$: components of points in $\mathcal{B}^n$

$$x^\alpha = \begin{cases} x, & \text{if } \alpha = 1, \\ \overline{x}, & \text{if } \alpha = 0. \end{cases}$$

$X \vee Y = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$

$X \wedge Y = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n) = (x_1 y_1, x_2 y_2, \ldots, x_n y_n)$

$\overline{X} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$

$X \leq Y$ (with $X, Y \in \mathcal{B}^n$) if and only if $x_i \leq y_i$ for $i = 1, 2, \ldots, n$

e_k : a unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ of appropriate dimension, with 1 in kth position

e_A: the characteristic vector of $A \subseteq \{1, 2, \ldots, n\}$, that is, $e_A = \sum_{k \in A} e_k$; $e_\emptyset = 0$.

$supp(X)$: the support of $X \in \mathcal{B}^n$, that is, the set $\{i \in \{1, 2, \ldots, n\} \mid x_i = 1\}$

$T_{A,B} = \{X \in \mathcal{B}^n \mid x_i = 1 \text{ for all } i \in A \text{ and } x_j = 0 \text{ for all } j \in B\}$

$f, g, h, \ldots$: Boolean functions

$\phi, \psi, \theta, \ldots$: Boolean expressions

$\mathbb{1}_n$: the function that takes constant value 1 on $\mathcal{B}^n$

$\mathbb{0}_n$: the function that takes constant value 0 on $\mathcal{B}^n$

$T(f)$: the set of true points of function f

$F(f)$: the set of false points of function f

$minT(f)$: the set of minimal true points of a positive function f

$maxF(f)$: the set of maximal false points of a positive function f

f^d : the dual of function f

$|\phi|$: the (encoding) length, or size, of a Boolean expression ϕ; when ϕ is a DNF, $|\phi|$ is simply the number of literals appearing in ϕ

$|f|$: for a positive function f, $|f|$ denotes the size of the complete (prime irredundant) DNF ϕ of f, that is, $|f| \stackrel{\text{def}}{=} |\phi|$

$||\phi||$: the number of terms of a DNF ϕ

$(\omega_1, \omega_2, \ldots, \omega_n, \omega)$: the Chow parameters of a Boolean function on $\mathcal{B}^n$

$(\pi_1, \pi_2, \ldots, \pi_n, \pi)$: the modified Chow parameters of a Boolean function on $\mathcal{B}^n$