Introduction

The Laplacian acting on functions of finitely many variables appeared in the works of Pierre Laplace (1749–1827) in 1782. After nearly a century and a half, the infinite-dimensional Laplacian was defined. In 1922 Paul Lévy (1886–1971) introduced the Laplacian for functions defined on infinite-dimensional spaces.

The infinite-dimensional analysis inspired by the book of Lévy *Leçons d'analyse fonctionnelle* [93] attracted the attention of many mathematicians. This attention was stimulated by the very interesting properties of the Lévy Laplacian (which often do not have finite-dimensional analogues) and its various applications.

In a work [68] (published posthumously in 1919) Gâteaux gave the definition of the mean value of the functional over a Hilbert sphere, obtained the formula for computation of the mean value for the integral functionals and formulated and solved (without explicit definition of the Laplacian) the Dirichlet problem for a sphere in a Hilbert space of functions. In this work he called harmonic those functionals which coincide with their mean values.

In a note written in 1919 [92], which complements the work of Gâteaux, Lévy gave the explicit definition of the Laplacian and described some of its characteristic properties for the functions defined on a Hilbert function space.

In 1922, in his book [93] and in another publication [94] Lévy gave the definition of the Laplacian for functions defined on infinite-dimensional spaces and described its specific features. Moreover he developed the theory of mean values and using the mean value over the Hilbert sphere, solved the Dirichlet problem for Laplace and Poisson equations for domains in a space of sequences and in a space of functions, obtained the general solution of a quasilinear equation. We have mentioned here only a few of a great number of results given in Lévy's book which is the classical work on infinite-dimensional analysis.

The second half of the twentieth century and the beginning of twenty-first century follows a period of development of a number of trends originated in [93], and the infinite-dimensional Laplacian has become an object of systematic study. This was promoted by the appearance of its second edition *Problèmes concrets d'analyse fonctionnelle* [95] in 1951 and the appearance, largely due to the initiative of Polishchuk, of its Russian translation (edited by Shilov) in 1967. During this period, there were published, among others, the works of: Lévy [96], Polishchuk [111–125], Feller [36–66], Shilov [132–135], Nemirovsky and Shilov [102], Nemirovsky [100, 101], Dorfman [28–33], Sikiryavyi [137–145], Averbukh, Smolyanov and Fomin [10], Kalinin [82], Sokolovsky [146–151], Bogdansky [13–22], Bogdansky and Dalecky [23], Naroditsky [99], Hida [75–78], Hida and Saito [79], Hida, Kuo, Potthoff and Streit [80], Yadrenko [158], Hasegawa [72–74], Kubo and Takenaka [85], Gromov and Milman [69], Milman [97, 98], Kuo [86–88], Kuo, Obata and Saito [89, 90], Saito [126–129], Saito and Tsoi [130], Obata [103–106], Accardi, Gibilisco and Volovich [4], Accardi, Roselli and Smolyanov [5], Accardi and Smolyanov [6], Accardi and Bogachev [1–3], Zhang [159], Koshkin [83, 84], Scarlatti [131], Arnaudon, Belopolskaya and Paycha [9], Chung, Ji and Saito [26], Léandre and Volovich [91], Albeverio, Belopolskaya and Feller [8].

Many problems of modern science lead to equations with Lévy Laplacians and Lévy–Laplace type operators. They appear, for example, in superconductivity theory [24, 71, 152, 155], the theory of control systems [121, 122], Gauss random field theory [158] and the theory of gauge fields (the Yang–Mills equation) [4], [91].

Lévy introduced the infinite-dimensional Laplacian acting on a function $U(x)$ by the formula

$$\Delta_L U(x_0) = 2 \lim_{\varrho \to 0} \frac{\mathfrak{M}_{(x_0,\varrho)} U(x) - U(x_0)}{\varrho^2}$$

(the Lévy Laplacian), where $\mathfrak{M}_{(x_0,\varrho)} U(x)$ is the mean value of the function $U(x)$ over the Hilbert sphere of radius ϱ with centre at the point x_0.

Given a function defined on the space of a countable number of variables we have

$$\Delta_L U(x_1, \ldots, x_n, \ldots) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\partial^2 U}{\partial x_k^2},$$

while for functions defined on a functional space we have

$$\Delta_L U(x(t)) = \frac{1}{b-a} \int_a^b \frac{\delta^2 U(x)}{\delta x(s)^2}\, ds,$$

where $\delta^2 U(x)/\delta x(s)^2$ is the second-order variational derivative of $U(x(t))$.

But already, in 1914, Volterra [154] had used different second-order differential expressions such as

$$\Delta_0 V(x(t)) = \int_a^b \frac{\delta^2 V(x)}{\delta x(s)\delta x(s)}\, ds$$

(the Volterra Laplacian), where $\delta^2 V(x)/\delta x(s)\delta x(\tau)$ is the second mixed variational derivative of $V(x(t))$. In 1966 Gross [70] and Dalecky [27] independently defined the infinite-dimensional elliptic operator of the second order which includes the Laplace operator

$$\Delta_0 V(x(t)) = \mathrm{Tr}\, V''(x),$$

where $V''(x)$ is the Hessian of the function $V(x)$ at the point x. For a function V defined on a functional space, $\Delta_0 V(x(t))$ is the Volterra Laplacian, and for functions defined on the space of a countable number of variables, we have

$$\Delta_0 V(x_1, \dots, x_n, \dots) = \sum_{k=1}^{\infty} \frac{\partial^2 V}{\partial x_k^2}.$$

There exists a number of other examples of second-order infinite-dimensional differential expressions which considerably differ from the differential expressions of Lévy type. The corresponding references can be found in the bibliography to the monographs of Berezansky and Kondratiev [12] and Dalecky and Fomin [27].

The present book deals with the problems of the theory of equations with the Lévy Laplacians and Lévy–Laplace operators. It is based on the author's papers [36–38, 40, 50–66] and the paper [8].

In Chapter 1 we give the definition of the Lévy Laplacian and describe some of its properties.

In the foreword to his book [95], Lévy wrote: 'In the theories which we mentioned, we essentially face the laws of great numbers similar to the laws of the theory of probabilities . . .'. The probabilistic treatment of the Lévy Laplacian in the second, third, and fourth chapters allows us to enlarge on a number of its interesting properties. Let us mention some of them. The Lévy Laplacian gives rise to operators of arbitrary order depending on the choice of the domain of definition of the operator. There is a huge number of harmonic functions of infinitely many variables connected with the Lévy Laplacian. The natural domain of definition of the Lévy Laplacian and that of the symmetrized Lévy Laplacian do not intersect. Starting from the non-symmetrized Lévy Laplacian, one can construct a symmetric and even a self-adjoint operator.

Problems in the theory of equations with Lévy Laplacians are considered in Chapters 5–7.

First, we concentrate our attention on the main classes of linear elliptic and parabolic equations with Lévy Laplacians.

The equations which describe real physical processes are, as a rule, nonlinear. The theory of linear equations with the Lévy Laplacian is quite developed (see the bibliography). On the other hand, the theory of nonlinear equations with the Lévy Laplacian has only recently begun to be developed. The final two chapters deal with elliptic quasilinear and nonlinear and parabolic nonlinear equations with the Lévy Laplacian.

We will see how striking is the difference (especially in the nonlinear case) between the theories of infinite-dimensional and n-dimensional partial differential equations.

Finally in the Appendix we apply the results of Chapter 3 to the construction of Dirichlet forms associated with the Lévy–Laplace operator, and show the connection between these forms and Markov processes.

There is no doubt that the reader of this book will see that the properties of the Lévy Laplacian, as a rule, have no analogues with the classical finite-dimensional Laplacian. Moreover, the differences are so essential that one can call them pathological if the properties of the Laplace operator for functions of a finite number of variables are considered to be the norm. However, from another point of view the opposite statement is true as well.

It should be emphasized that in this book we consider only the Lévy Laplacian. We do not consider here the problems of the theory of equations and operators of Lévy type (which naturally generalize the equations with Lévy Laplacians and Lévy–Laplace operators) considered in our papers [39, 41–49].

Unfortunately, a lot of the results concerning different trends originated in the book by Lévy are not included in this work although they undoubtedly deserve to be considered. In particular we do not discuss here the well-known approach to the Lévy Laplacian via white noise theory [80, 88]. I hope that this is compensated for to some extent by the large bibliography presented here.

With great warmth I recollect numerous conversations on the topics discussed in this book with those who have departed: Yu. L. Dalecky (1926–1997), O. A. Ladyzhenskaya (1922–2004), E. M. Polishchuk (1914–1987) and G. E. Shilov (1917–1975).

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1
The Lévy Laplacian

1.1 Definition of the infinite-dimensional Laplacian

Let H be a countably-dimensional real Hilbert space. Consider a scalar function $F(x)$ on H, where $x \in H$.

Lévy introduced the infinite-dimensional differential Laplacian by

$$\Delta_L F(x_0) = 2 \lim_{\varrho \to 0} \frac{\mathfrak{M}_{(x_0,\varrho)} F(x) - F(x_0)}{\varrho^2}. \tag{1.1}$$

This definition assumes that $F(x)$ has the mean value $\mathfrak{M}_{(x_0,\varrho)} F(x)$, for $\varrho < \varrho_0$, and that the limit at the right-hand side of (1.1) exists.

We define the mean value of the function $F(x)$ over the Hilbert sphere $\|x - x_0\|_H^2 = \varrho^2$ as the limit (if it exists) of the mean value, over the n-dimensional sphere, of the function $F(\sum_{k=1}^n x_k f_k) = f(x_1, \dots, x_n)$, i.e., of the restriction of the function $F(x)$ on the n-dimensional subspace with the basis $\{f_k\}_1^n$, $\quad x_k = (x, f_k)_H$:

$$\mathfrak{M}_{(x_0,\varrho)} F(x) = \lim_{n \to \infty} M_n F(x),$$

$$M_n F(x) = \frac{1}{s_n} \int\limits_{\sum_{k=1}^n (x_k - x_{0k})^2 = \varrho^2} f(x_1, \dots, x_n) d\sigma_n,$$

where s_n is the area, and $d\sigma_n$ is the element of the n-dimensional sphere surface. In general, the mean value depends on the choice of the basis.

It follows immediately from its definition that the mean value is additive and homogeneous: if there exists $\mathfrak{M}_{(x_o,\varrho)} F_k$, $k = 1, \dots, m$, then there exists

$$\mathfrak{M}_{(x_o,\varrho)} \Big(\sum_{k=1}^m c_k F_k \Big) = \sum_{k=1}^m c_k \mathfrak{M}_{(x_0,\varrho)} F_k.$$

The mean value possesses the multiplicative property: if there exists $\mathfrak{M}_{(x_o,\varrho)} F_k$, and the F_k are uniformly continuous in a bounded domain $\Omega \in H$, which contains the sphere $\|x - x_0\|_H^2 = \varrho^2$, then there exists

$$\mathfrak{M}_{(x_0,\varrho)}\Bigl(\prod_{k=1}^{m} F_k\Bigr) = \prod_{k=1}^{m} \mathfrak{M}_{(x_0,\varrho)} F_k.$$

This property follows from the following statement of Lévy. Let function $F(x)$ be uniformly continuous on the sphere $\|x - x_0\|_H^2 = \varrho^2$, and let the average of the function $F(x)$ exist (i.e., $M_n \to M$, $M = \mathfrak{M}_{(x_0,\varrho)} F$). Then for each $\delta > 0$ we have

$$\lim_{n\to\infty} \frac{1}{s_n} m_n\{x : |f(x_1, \dots, x_n) - M| > \delta\} = 0$$

(here m_n denotes the Lebesgue measure).

Note that the definition of the Laplacian via mean values is valid for the finite-dimensional case as well.

The definition (1.1) does not assume differentiability of the function $F(x)$. However, if the function $F(x)$ is twice strongly differentiable, then the following representation of the Lévy Laplacian holds.

Lemma 1.1 *Let the function $F(x)$ be twice strongly differentiable in point x_0, and the Laplacian Δ_L exist. Then*

$$\Delta_L F(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (F''(x_0) f_k, f_k)_H, \tag{1.2}$$

where $F''(x_0)$ is the Hessian of the function $F(x)$ in the point x_0, $F''(x_0) \in \{H \to H\}$, and $\{f_k\}_1^\infty$ is some chosen orthonormal basis in H.

Indeed, it follows from the definition of the mean value that $\mathfrak{M}_{(x_0,\varrho)} F(x) = \mathfrak{M} F(x_0 + \varrho h)$, where $\mathfrak{M}\Phi(h)$ is the mean value of the function Φ over the sphere $\|h\|_H^2 = 1$. Therefore, taking into account that for $a \in H$ $\mathfrak{M}\,(a, h)_H = 0$, because $\frac{1}{s_n} \int_{\sum_{k=1}^n h_k^2 = 1} h_k d\sigma_n = 0$, we have

$$\begin{aligned}
\frac{1}{\varrho^2}\{\mathfrak{M}_{(x_0,\varrho)} F(x) - F(x_0)\} &= \frac{1}{\varrho^2}\{\mathfrak{M} F(x_0 + \varrho h) - F(x_0)\} \\
&= \frac{1}{\varrho^2}\left\{\mathfrak{M}\left[(F'(x_0), \varrho h)_H + \frac{1}{2}(F''(x_0)\varrho h, \varrho h)_H + r(x_0, \varrho h)\right]\right\} \\
&= \frac{1}{\varrho^2}\left\{\overline{\lim_{n\to\infty}} M_n \left[\frac{\varrho^2}{2}(F''(x_0)h, h)_H + r(x_0, \varrho h)\right]\right\} \\
&\geq \frac{1}{2}\overline{\lim_{n\to\infty}} M_n (F''(x_0)h, h)_H \underline{\lim}_{n\to\infty} \frac{M_n r(x_0, \varrho h)}{\varrho^2} \\
&\qquad \left(\text{and } \frac{r(x_0, \varrho h)}{\|\varrho h\|_H^2} \to 0 \quad \text{as} \quad \|\varrho h\|_H^2 \to 0\right);
\end{aligned}$$

similarly we have

$$\frac{1}{\varrho^2}\{\mathfrak{M}_{(x_0,\varrho)}F(x)-F(x_0)\} \le \frac{1}{2}\underline{\lim}_{n\to\infty} M_n(F''(x_0)h,h)_H + \overline{\lim_{n\to\infty}}\frac{M_n r(x_0,\varrho h)}{\varrho^2}.$$

From this we obtain

$$\frac{1}{\varrho^2}\{\mathfrak{M}_{(x_0,\varrho)}F(x)-F(x_0)\} - \varepsilon(\varrho) \le \frac{1}{2}\underline{\lim}_{n\to\infty} M_n(F''(x_0)h,h)_H$$
$$\le \frac{1}{2}\overline{\lim_{n\to\infty}} M_n(F''(x_0)h,h))_H \le \frac{1}{\varrho^2}\{\mathfrak{M}_{(x_0,\varrho)}F(x)-F(x_0)\} + \varepsilon(\varrho),$$

where $\varepsilon(\varrho) = \frac{1}{\varrho^2}\sup_{\|h\|_H^2=1}|r(x_0,\varrho h)|$, $\varepsilon(\varrho)\to 0$ as $\varrho\to 0$.

Therefore, $\Delta_L F(x_0) = \mathfrak{M}(F''(x_0)h,h)_H$.

Taking into consideration that, according to formula of Ostrogradsky,

$$M_n h_k^2 = \frac{1}{s_n}\int\limits_{\sum_{k=1}^n h_k^2=1} h_k^2\, d\sigma_n = \frac{1}{s_n}\int\limits_{\sum_{k=1}^n h_k^2\le 1} \frac{\partial h_k}{\partial h_k} dh_1\dots dh_n$$
$$= \frac{v_n}{s_n} = \frac{\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})} = \frac{1}{n},$$
$$M_n h_k h_j = \frac{1}{s_n}\int\limits_{\sum_{k=1}^n h_k^2=1} h_k h_j\, d\sigma_n = 0 \quad \text{for} \quad j\ne k,$$

(here $h_k = (h, f_k)_H$, v_n is volume, s_n is the area of surface of the sphere $\sum_{k=1}^n h_k^2 = 1$, $\Gamma(s)$ the gamma function), we obtain that

$$\Delta_L F(x_0) = \mathfrak{M}(F''(x_0)h,h)_H = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n (F''(x_0)f_k, f_k)_H.$$

□

If at the given point x_0 the function $F(x)$ is twice differentiable only with respect to the subspace Y of the space H (i.e., the second differential of the function $F(x)$ at the point x_0 does not exist for all increments $h\in H$, but $d^2F(x_0,y) = (F_Y''(x_0)y,y)_H$ exists for the increments y that form the subspace Y of the space H, and the second derivative of the function $F(x)$ at the point x_0 with respect to the subspace Y is the operator $F_Y''(x_0)\in\{Y\to Y'\}$, where Y' is the space conjugate to Y), then from (1.1) we deduce that

$$\Delta_L F(x_0) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n (F_Y''(x_0)f_k, f_k)_H, \tag{1.3}$$

provided that the basis $\{f_k\}_1^\infty$ is orthonormal in H and that $f_k\in Y$.

Now we give the formula for the infinite-dimensional Laplacian obtained by Lévy.

Let there be a function

$$F(x) = f(U_1(x), \dots, U_m(x)),$$

where $f(u_1, \dots, u_m)$ is a twice continuously differentiable function of m variables in the domain of values $\{U_1(x), \dots, U_m(x)\}$ in $\mathbb{R}^m$, $U_j(x)$ are some twice strongly differentiable functions, and the $\Delta_L U_j(x)$ exist ($j = 1, \dots, m$). Then $\Delta_L F(x)$ exists, and

$$\Delta_L F(x) = \sum_{j=1}^{m} \frac{\partial f}{\partial u_j}\Big|_{u_j=U_j(x)} \Delta_L U_j(x). \tag{1.4}$$

Indeed, the second differential of the function $F(x)$ at the point x for increment $h \in H$ is

$$d^2F(x;h) = (F''(x)h, h)_H = \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial u_i \partial u_j}\Big|_{u_l=U_l(x)} (U_i'(x), h)_H (U_j'(x), h)_H$$
$$+ \sum_{j=1}^{m} \frac{\partial f}{\partial u_j}\Big|_{u_j=U_j(x)} (U_j''(x)h, h)_H.$$

According to (1.2),

$$\Delta_L F(x) = \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial u_i \partial u_j}\Big|_{u_l=U_l(x)} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (U_i'(x), f_k)_H (U_j'(x), f_k)_H$$
$$+ \sum_{j=1}^{m} \frac{\partial f}{\partial u_j}\Big|_{u_j=U_j(x)} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (U_j''(x) f_k, f_k)_H.$$

But

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (U_i'(x), f_k)_H (U_j'(x), f_k)_H = 0,$$

(because $(U_l'(x), f_k)_H \to 0$ as $k \to \infty$), and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (U_j''(x) f_k, f_k)_H = \Delta_L U_j(x).$$

Therefore,

$$\Delta_L F(x) = \sum_{j=1}^{m} \frac{\partial f}{\partial u_j}\Big|_{u_j=U_j(x)} \Delta_L U_j(x).$$

A series of consequences follows from formula (1.4).

1. If the functions $U_k(x)$ are harmonic in some domain Ω, $k = 1, \dots, m$, then the function $F(x)$ also is harmonic in Ω.

2. The Lévy Laplacian is a 'differentiation'. It is enough to set $F(x) = U_1(x)U_2(x)$: then

$$\Delta_L[U_1(x)U_2(x)] = \Delta_L U_1(x) \cdot U_2(x) + U_1(x) \cdot \Delta_L U_2(x).$$

3. The Liouville theorem does not hold for harmonic functions of an infinite number of variables, i.e., there exists a function that is not equal to a constant which is harmonic and bounded in the whole space: it is sufficient to put $F(x) = f(U(x))$, where $f(u)$ is a differentiable function in $\mathbb{R}^1$ bounded together with its derivative, $U(x)$, which is a harmonic function in the whole of H. For example, $F(x) = \cos(\alpha, x)_H,\ \alpha \in H$.

1.2 Examples of Laplacians for functions on infinite-dimensional spaces

For functions on a space of sequences, the Lévy Laplacian is an operator with an infinite number of partial derivatives, and for functions on spaces of functions of finitely many variables, the Lévy Laplacian is an operator in variational derivatives.

Example 1.1 Let $H = l_2$ be the space of sequences $\{x_1, \dots, x_n, \dots\}$ such that $\sum_{k=1}^{\infty} x_k^2 < \infty$.

If the function $F(x_1, \dots, x_n, \dots)$ is twice strongly differentiable, and the Laplacian exists, then its Hessian is the matrix

$$\left\| \frac{\partial^2 F(x)}{\partial x_i \partial x_k} \right\|_{i,k=1}^{\infty}$$

which induces a bounded operator in l_2 :

$$(F''(x)h, h)_{l_2} = \sum_{i,k=1}^{\infty} \frac{\partial^2 F(x)}{\partial x_i \partial x_k} h_i h_k,$$

and (1.2) yields that

$$\Delta_L F(x_1, \dots, x_n, \dots) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\partial^2 F(x)}{\partial x_k^2}.$$

Example 1.2 Let $H = L_2(0, 1)$ be the space of functions $x(t)$, square integrable on $[0, 1]$.

If the second differential of the twice differentiable function $F(x(t))$ has the form

$$d^2 F(x; h) = \int_0^1 \frac{\delta^2 F(x)}{\delta x(s)^2} h^2(s)\, ds + \int_0^1\int_0^1 \frac{\delta^2 F(x)}{\delta x(s)\delta x(\tau)} h(s)h(\tau)\, ds d\tau,$$

where the second variational derivative $\delta^2 F(x)/\delta x(s)^2$ and the second mixed variational derivative $\delta^2 F(x)/\delta x(s)\delta x(\tau)$ are continuous with respect to s and s, τ respectively (here $h(t) \in L_2(0, 1)$), then one says that $d^2 F(x; h)$ has normal form [95], and if

$$d^2 F(x; h) = \int_0^1 \int_0^1 \frac{\delta^2 F(x)}{\delta x(s)\delta x(\tau)} h(s)h(\tau)\, ds d\tau,$$

than one says that it has regular form [154].

We denote by $\mathcal{B}$ the set of all uniformly dense (according to the Lévy terminology) bases in $L_2(0, 1)$, i.e. orthonormal bases $\{f_k\}_1^\infty$ in $L_2(0, 1)$, such that

$$\lim_{n\to\infty} (y, \varphi_n)_{L_2(0,1)} = (y, 1)_{L_2(0,1)} \qquad \text{for all } y \in L_2(0, 1),$$

where $\varphi_n(s) = \frac{1}{n}\sum_{k=1}^n f_k^2(s)$.

As has been shown by Polishchuk (in his comments to the Russian translation of [95]), all orthonormal bases which are the eigenfunctions of some Sturm–Liouville problem are uniformly dense.

Let the function $F(x)$ be twice strongly differentiable, and the second differential have normal form. Then

$$\Delta_L F(x(t)) = \int_0^1 \frac{\delta^2 F(x)}{\delta x(s)^2}\, ds$$

for arbitrary basis from $\mathcal{B}$.

Indeed,

$$(F''(x)h, h)_{L_2(0,1)} = \int_0^1 \int_0^1 \left[\delta(s-\tau)\frac{\delta^2 F(x)}{\delta x(s)^2}\, ds + \frac{\delta^2 F(x)}{\delta x(s)\delta x(\tau)}\right] h(s)h(\tau)\, ds d\tau$$

($\delta(s-\tau)$ is the delta function), and, according to (1.2),

$$\Delta_L F(x(t)) = \lim_{n\to\infty}\left[\int_0^1 \frac{\delta^2 F(x)}{\delta x(s)^2}\varphi_n(s)\, ds + \frac{\delta^2 F(x)}{\delta x(s)\delta x(\tau)}\psi_n(s, \tau)\, ds d\tau\right],$$

where $\psi_n(s, \tau) = \frac{1}{n}\sum_{k=1}^n f_k(s) f_k(\tau)$.

But

$$\int_0^1 \int_0^1 \frac{\delta^2 F(x)}{\delta x(s)\delta x(\tau)}\psi_n(s, \tau)\, ds d\tau \to 0 \quad \text{as} \quad n \to \infty,$$