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1 HEIGHTS

1.1. Introduction

This chapter contains preliminary material on absolute values and the elementary theory of heights on projective varieties. Most of this material is quite standard, although we have included some of the finer results on classical heights which are not usually treated in other texts.

In Section 1.2 we start with absolute values, and places are introduced as equivalence classes of absolute values. The definitions of residue degree and ramification index are given, as well as their basic properties and behaviour with respect to finite degree extensions. In Sections 1.3 and 1.4 we introduce normalized absolute values and the all-important product formula in number fields and function fields. Section 1.5 contains the definition of the absolute Weil height in projective spaces, the characterization of points with height 0, and a general form of Liouville's inequality in diophantine approximation. Section 1.6 studies the height of polynomials and Mahler's measure and proves Gauss's lemma and its counterpart at infinity, Gelfond's lemma. Section 1.7, which can be omitted in a first reading, elaborates further on various comparison results about heights and norms of polynomials, including an interesting result of Per Enflo on ℓ_1-norms.

The presentation of the material in this chapter is self contained with the exception of Section 1.2, where the basic facts about absolute values are quoted from standard reference books (N. Bourbaki [**47**], Ch.VI, S. Lang [**173**], Ch.XII, and N. Jacobson [**157**], Ch.IX).

1.2. Absolute values

Definition 1.2.1. *An* ***absolute value*** *on a field K is a real valued function $|\ |$ on K such that:*

(a) $|x| \geq 0$ *and* $|x| = 0$ *if and only if* $x = 0$.

(b) $|xy| = |x|\ |y|$.

(c) $|x + y| \leq |x| + |y|$ *(triangle inequality).*

1.2.2. The **trivial absolute value** is equal to 1 except at 0. If an absolute value satisfies instead of the triangle inequality (c) the stronger condition

(c′) $|x+y| \le \max\{|x|, |y|\}$,

then it is called **non-archimedean**. If (c′) fails to hold for some $x, y \in K$, then the absolute value is called **archimedean**. The distance of $x, y \in K$ is $|x-y|$. This metric induces a topology on K. In the non-archimedean case, we have an ultrametric distance and (c′) is called the **ultrametric triangle inequality** . If two absolute values define the same topology, they are called **equivalent** .

Proposition 1.2.3. *Two absolute values $|\ |_1$, $|\ |_2$ are equivalent if and only if there is a positive real number s such that*

$$|x|_1 = |x|_2^s$$

for $x \in K$.

Proof: See [**157**], Th.9.1 or [**173**], Prop.XII.1.1. □

1.2.4. A **place** v is an equivalence class of non-trivial absolute values. By $|\ |_v$ we denote an absolute value in the equivalence class determined by the place v. If the field L is an extension of K and v is a place of K, we write $w|v$ for a place w of L if and only if the restriction to K of any representative of w is a representative of v, and say that w extends v and, equivalently, that w lies over v. We also employ the notation $w|v$ (that is, w divides v), motivated by the fact that non-archimedean places in number fields correspond to prime ideals.

The **completion** of K with respect to the place v is an extension field K_v with a place w such that:

(a) $w|v$.

(b) The topology of K_v induced by w is complete.

(c) K is a dense subset of K_v in the above topology.

The completion exists and is unique up to isometric isomorphisms ([**157**], Th.9.7 or [**173**], Prop.XII.2.1). By abuse of notation, we shall denote the unique place w also by v.

Example 1.2.5. If the field is $\mathbb{Q}$, then there is only one archimedean place ∞ on $\mathbb{Q}$, given by the ordinary absolute value $|\ |$. We also write $|\ |_\infty$ for this absolute value (cf. [**157**], Th.9.4).

For a prime p we have the ***p*-adic absolute value** $|\ |_p$ determined as follows. Let $m/n \in \mathbb{Q}$ be a rational number and write it in the form

$$\frac{m}{n} = p^a \frac{m'}{n'},$$

where m', n' are integers coprime with p. Then we set

$$\left|\frac{m}{n}\right|_p = p^{-a}.$$

In fact, it suffices to define $|\ |_p$ by the conditions

$$|q|_p = \begin{cases} 1 & \text{for primes } q \neq p \\ \frac{1}{p} & \text{if } q = p. \end{cases}$$

The p-adic absolute values so defined give us a set of inequivalent representatives for all non-archimedean places on $\mathbb{Q}$ ([**157**], Th.9.5). The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the place p. The compact subset $\mathbb{Z}_p$ of p-adic integers is the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$ (for compactness, see [**47**], Ch.VI, §5, no.1, Prop.2). On the other hand, the completion of $\mathbb{Q}$ with respect to the archimedean place ∞ is $\mathbb{R}$. In full generality, we have the following well-known **Theorem of Ostrowski**:

Theorem 1.2.6. *The only complete archimedean fields are* $\mathbb{R}$ *and* $\mathbb{C}$.

Proof: [**157**], §9.5 or [**47**], Ch.VI, §6, no.4, Th.2. □

Proposition 1.2.7. *Let K be a field which is complete relative to an absolute value $|\ |_v$ and let L be a finite-dimensional extension field of K. Then there is a unique extension of $|\ |_v$ to an absolute value $|\ |_w$ of L. For any $x \in L$ the equation*

$$|x|_w = |N_{L/K}(x)|_v^{1/[L:K]}$$

holds, where $N_{L/K}$ is the norm from L to K. Moreover, the field L is complete with respect to $|\ |_w$.

Proof: [**157**], Th.s 9.8, 9.9, 9.12 or [**47**], Ch.VI, §8, no.7, Prop.10. □

Remark 1.2.8. Clearly, the preceding proposition implies that there is a unique extension to an absolute value on the algebraic closure of K. Note however that the last clause of this proposition need not hold for infinite-dimensional extensions; a well-known example is an algebraic closure of the p-adic field $\mathbb{Q}_p$ (cf. S. Bosch, U. Güntzer, and R. Remmert [**43**], 3.4.3).

1.2.9. Let K be a field with a non-archimedean place v and let L be a finite-dimensional field extension of K. Assume that w is a place of L with $w|v$. The ring

$$R_v := \{x \in K \mid |x|_v \leq 1\}$$

is called the **valuation ring** of v. The definition is obviously independent of the representative $|\ |_v$ of v. R_v is a local ring with unique maximal ideal $\mathfrak{m}_v := \{x \in K \mid |x|_v < 1\}$.

The **residue field** $k(v)$ is defined by $R_v/\mathfrak{m}_v$. The quotient map $R_v \to k(v), x \mapsto \overline{x}$ is called the **reduction**. Applying it to coefficients, it extends to polynomials and to power series.

The **residue degree** $f_{w/v}$ of L/K in w is the dimension of $k(w)$ over $k(v)$. Let $|\ |_w$ be an absolute value representing w and $|\ |_v$ the restriction of $|\ |_w$ to K. The **value group** $|K^\times|_v$ is a multiplicative subgroup of $|L^\times|_w$ and its index is called the **ramification index** $e_{w/v}$ of w in v.

The place v is called **discrete** if the value group $|K^\times|_v$ is cyclic. Then $\mathfrak{m}_v$ is a principal ideal and any principal generator is called a **local parameter**.

The following result is the very useful **Hensel's lemma**.

Lemma 1.2.10. *Let K be complete with respect to a non-archimedean place v. Let $f(t) \in K[t]$ be a monic polynomial with reduction $\overline{f}(t) = \overline{g}(t)\overline{h}(t)$ for some monic coprime polynomials $\overline{g}(t), \overline{h}(t) \in k(v)[t]$. Then there exist monic polynomials $G(t), H(t) \in R_v[t]$ with $F(t) = G(t)H(t)$ and $\overline{G}(t) = \overline{g}(t)$, $\overline{H}(t) = \overline{h}(t)$.*

Proof: For discrete valuations, we refer to [**157**], §9.11. The general case is proved in [**43**], 3.3.4. □

Proposition 1.2.11. *Let L/K and $w|v$ be as in 1.2.9.*

(a) *The residue degree and the ramification index do not change if we pass to completions.*

(b) *The product of the residue degree and the ramification index is at most $[L:K]$, with equality if v is discrete and K is complete relative to v.*

Proof: [**157**], §9.10 or [**173**], Prop.s XII.4.2, XII.6.1, and §XII.5. □

1.2.12. A **number field** is a finite-dimensional field extension of $\mathbb{Q}$. The ring of algebraic integers of K is denoted by O_K. Now let L be a locally compact field containing a number field K as a dense subset and assume that the topology is not discrete. Then it follows that L is complete because it is locally compact. The classification of non-discrete locally compact fields is well known and tells us that there is a place v of K such that L is the completion of K with respect to v. Moreover, if L is connected, then L is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or to a finite extension of $\mathbb{Q}_p$. The closure of O_K in L coincides with the valuation ring R_v of L.

On the other hand, every completion of a number field with respect to a non-archimedean place is a finite extension of $\mathbb{Q}_p$, hence locally compact. For details, we refer to [**47**], Ch.VI, §9, no.3, Th.1. The following result of Artin and Whaples is called the **approximation theorem**:

Theorem 1.2.13. *Let $|\ |_1, \dots, |\ |_n$ be inequivalent non-trivial absolute values on a field K. Then for $x_1, \dots, x_n \in K$ and $\varepsilon > 0$ there is $x \in K$ such that*

$$|x - x_k|_k < \varepsilon \quad (k = 1, \dots, n).$$

Proof: [**157**], §9.2 or [**173**], Th.XII.1.2. □

1.3. Finite-dimensional extensions

Let K be a field with a fixed non-trivial absolute value $|\ |_v$.

Proposition 1.3.1. *Let L be a finite-dimensional field extension of K generated by a single element ξ. If $f(t)$ is the monic minimal polynomial of ξ over K and*

$$f(t) = f_1^{k_1}(t) \cdots f_r^{k_r}(t)$$

is its decomposition into different irreducible monic factors $f_j(t) \in K_v[t]$, then for each j there is an injective homomorphism

$$\iota : L \longrightarrow K_j := K_v[t]/(f_j(t))$$

of field extensions over K, given by $\xi \mapsto t$. There is a unique extension $|\ |_j$ of the absolute value of K_v to K_j. The absolute values $|\ |_j$ are pairwise inequivalent. Moreover, K_j is the completion of L with respect to $|\ |_j$ and the embedding ι. For any absolute value $|\ |_w$ extending $|\ |_v$ to L, there is a unique $j \in \{1, \ldots, r\}$ such that the restriction of $|\ |_j$ to L is equal to $|\ |_w$.

Proof: Proposition 1.2.7 leads to the unique extension $|\ |_j$ of the absolute value of K_v. The map ι is a well-defined homomorphism of field extensions over K and the image of ι is dense in K_j. Hence K_j is the completion of L with respect to $|\ |_j$. If the restrictions of $|\ |_j$ and $|\ |_k$ are equivalent, then we have an isometric isomorphism of K_j onto K_k, leaving K_v fixed. Therefore, the images of ξ have to be roots of the same irreducible factor of $f(t)$ in $K_v[t]$, yielding $j = k$. Let $|\ |_w$ be an absolute value on L extending $|\ |_v$. The closure of K in L_w can be identified with K_v. Now ξ generates a finite-dimensional subfield of L_w over K_v which is complete by Proposition 1.2.7, therefore this subfield is L_w itself. Also ξ must be a root of some f_j, hence L_w is isomorphic to K_j over K_v. Moreover, we can assume that L is fixed under this isomorphism. Then it is clear from Proposition 1.2.7 that $|\ |_w$ is equal to the restriction of $|\ |_j$ to L. □

Corollary 1.3.2. *If L is a finite-dimensional separable field extension of K, then*

$$\sum_{w|v} [L_w : K_v] = [L : K],$$

where the sum ranges over all places w of L with $w|v$.

Proof: By the primitive element theorem (see N. Jacobson [**156**], §4.14), there is an element ξ of L which generates L over K. Proposition 1.3.1 implies the formula. □

Remark 1.3.3. If the extension is not separable, we still have $\sum_{w|v}[L_w : K_v] \leq [L : K]$. If L is generated by a single element over K, this is clear from Proposition 1.3.1. For the general case, we use induction on the degree.

Definition 1.3.4. *The number $[L_w : K_v]$ is called the **local degree** of L/K in w.*

Corollary 1.3.5. *Let L/K be a finite-dimensional Galois extension with Galois group $G = \mathrm{Gal}(L/K)$ and let $|\ |_{w_0}$, $|\ |_w$ be absolute values on L extending $|\ |_v$. Then there is an element $\sigma \in G$ with*

$$|x|_w = |\sigma(x)|_{w_0} \quad \text{for} \quad x \in L.$$

The completions L_w and L_{w_0} are isomorphic over K_v. However, they need not to be isomorphic over L.

Proof: As in the proof of Corollary 1.3.2, there is an element ξ of L with $L = K(\xi)$. If $f(t)$ is the minimal polynomial of ξ over K, then L_w is obtained by adjoining a root of $f_j(t)$ to K_v in a fixed splitting field of f over K_v, where $f_j(t)$ is an irreducible factor of $f(t)$ in $K_v[t]$. Since L is a Galois extension, all roots of f are contained in L_w, therefore $L_w = L_{w_0}$ as a field. Then the absolute values $|\ |_{w_0}$ and $|\ |_w$ correspond to embeddings ι_0 and ι of L into L_{w_0} over K. There is a unique $\rho \in \mathrm{Gal}(L_{w_0}/K_v)$ with $\iota = \rho \circ \iota_0$, given by $\iota_0(\xi) \mapsto \iota(\xi)$. If $|\ |$ is the unique absolute value of L_{w_0} extending the one of K_v and if σ is the unique element of G with $\rho \circ \iota_0 = \iota_0 \circ \sigma$, then

$$|x|_w = |\iota(x)| = |\rho \circ \iota_0(x)| = |\iota_0 \circ \sigma(x)| = |\sigma(x)|_{w_0} \quad \text{for } x \in L. \qquad \square$$

1.3.6. Let K be a field with a fixed non-trivial absolute value $|\ |_v$. We consider a finite-dimensional separable extension field L of K and a place w of L with $w|v$. For any $x \in L$, we define

$$\|x\|_w := |N_{L_w/K_v}(x)|_v$$

and

$$|x|_w := |N_{L_w/K_v}(x)|_v^{1/[L:K]}.$$

We know from Proposition 1.2.7 that the restriction of $|N_{L_w/K_v}|_v^{1/[L_w:K_v]}$ to L is a representative of w extending $|\ |_v$. The obvious inequality $[L_w : K_v] \le [L : K]$ implies that $|\ |_w$ is an absolute value representing w. If v is not archimedean or $[L_w : K_v] = 1$, we have that $\|\ \|_w$ is also an absolute value representing w. On the other hand, if v is archimedean and $[L_w : K_v] = 2$, we have $L_w = \mathbb{C}$ and $K_v = \mathbb{R}$. Assume that the restriction of $|\ |_v$ to $\mathbb{Q}$ is the ordinary absolute value; then $\|\ \|_w$ is not an absolute value because the triangle inequality is not satisfied.

Lemma 1.3.7. *Let $x \in K \setminus \{0\}$ and $y \in L \setminus \{0\}$. With the notation above*

$$\sum_{w|v} \log |x|_w = \log |x|_v\,,$$

$$\sum_{w|v} \log \|y\|_w = \log |N_{L/K}(y)|_v.$$

Proof: Corollary 1.3.2 implies the first statement. There is an element ξ of L with $L = K(\xi)$. With the notation of Proposition 1.3.1, we have $k_1 = \cdots = k_r = 1$ and an isomorphism

$$L \otimes_K K_v \xrightarrow{\sim} \prod_{j=1}^{r} K_v[t]/(f_j(t))$$

of K_v-algebras, given by $\xi \longrightarrow (t)_{j=1,\ldots,r}$ (this is a form of the Chinese remainder theorem). By Proposition 1.3.1 we get

$$N_{L/K}(y) = \prod_{w|v} N_{L_w/K_v}(y),$$

proving the second claim. □

1.3.8. If K is a number field, the archimedean absolute values of K are determined by the embeddings $\sigma : K \longrightarrow \mathbb{C}$ of K into the complex numbers. There are exactly $[K : \mathbb{Q}]$ such embeddings. An embedding σ is said to be real if $\sigma(K)$ is in the real subfield $\mathbb{R}$ of $\mathbb{C}$, and complex otherwise. If σ is a complex embedding, composition with complex conjugation yields a conjugate embedding $\overline{\sigma}$, and it is clear that σ and $\overline{\sigma}$ determine the same archimedean absolute value. Conversely, if σ and σ' are two embeddings of K in $\mathbb{C}$ determining the same absolute value, we have $\sigma' = \sigma$ or $\sigma' = \overline{\sigma}$. All this is immediate from Proposition 1.3.1, because K has a primitive element over $\mathbb{Q}$.

The completion of K at an archimedean place is isometric to either $\mathbb{R}$ or $\mathbb{C}$. Accordingly, the set of archimedean places is subdivided into **real places** and **complex places**.

Example 1.3.9. Let p be an odd prime and $K = \mathbb{Q}(\zeta)$ with ζ a primitive pth root of unity. Our goal in this example is to determine all extensions of an absolute value of $\mathbb{Q}$ to K.

The minimal polynomial of ζ over $\mathbb{Q}$ is given by

$$f(t) = t^{p-1} + t^{p-2} + \cdots + 1,$$

which is proved by applying Eisenstein's criterion to $f(t+1)$.

To begin with, we determine the extensions of the ordinary absolute value $|\ |_\infty$ of $\mathbb{Q}$ to K. The irreducible factors of $f(t)$ in $\mathbb{R}[t]$ have degree 2. By Proposition 1.3.1, there are exactly $\frac{p-1}{2}$ extensions of $|\ |_\infty$; all archimedean absolute values of K are associated to the $(p-1)/2$ pairs of complex conjugate embeddings of K and the local degree is equal to 2.

Next, we consider the extensions of the non-archimedean absolute value associated to a prime number q. Suppose first that $q \neq p$. We need to decompose $f(t)$ into irreducible factors over $\mathbb{Q}_q$. There is a smallest number $r \geq 1$ with $p|q^r - 1$, determined by the property that $\mathbb{F}_{q^r}$ is the smallest field of characteristic q containing a non-trivial pth root of unity (note also that, by Fermat's little theorem, $r|p-1$). In that case, the field $\mathbb{F}_{q^r}$

contains all pth roots of unity. Hence $f(t)$ is a product of $\frac{p-1}{r}$ distinct irreducible factors in $\mathbb{F}_q[t]$, each degree r. The same is true in $\mathbb{Q}_q[t]$ by Hensel's lemma (see Lemma 1.2.10). We conclude again by Proposition 1.3.1 that there are exactly $\frac{p-1}{r}$ extensions of $|\ |_q$ to K and the local degrees are equal to r. It is obvious that ζ remains a unit in any completion of K with respect to such an absolute value and its representative in the residue field is also a non-trivial primitive pth root of unity. By Proposition 1.2.11, the residue degree is equal to r and the ramification index is 1.

It remains to consider the place p. As before, Eisenstein's criterion shows that the polynomial $f(t)$ is irreducible in $\mathbb{Q}_p[t]$. Then there is only one extension $|\ |_v$ of $|\ |_p$ to K and the local degree is $p-1$. The minimal polynomial of $\zeta - 1$ over $\mathbb{Q}_p$ is $f(t+1)$ and so $N_{K_v/\mathbb{Q}_p}(\zeta - 1) = p$. Proposition 1.2.7 implies

$$|\zeta - 1|_v = p^{-1/(p-1)}.$$

By Proposition 1.2.11, the ramification index is equal to $p-1$ and the residue degree is equal to 1.

1.3.10. In the final part of this section, we handle finite-dimensional field extensions without separability assumptions. It turns out that it suffices to adjust the exponents in the normalization 1.3.6. Since we focus almost exclusively on number fields, the reader may skip the rest of this section in a first reading.

Let K be a field with absolute values $|\ |_v$ and let L/K be a finite-dimensional field extension. Our goal is to generalize Proposition 1.3.1 describing the extensions of $|\ |_v$ to the field L.

Since $L \otimes_K K_v$ is a finite-dimensional K_v-algebra, the structure theorem of commutative artinian rings ([**157**], Th.7.13) gives uniquely determined ideals R_j, which are local K_v-algebras with maximal ideals $\mathfrak{m}_j$ and such that

$$L \otimes_K K_v = \prod_{j=1}^{r} R_j. \tag{1.1}$$

We have natural embeddings of L and K_v into the residue field $K_j = R_j/\mathfrak{m}_j$ of R_j. By Proposition 1.2.7, there is a unique extension $|\ |_j$ of $|\ |_v$ to K_j. Clearly, L is dense in K_j, whence K_j is the completion of L with respect to this absolute value.

Proposition 1.3.11. *The restrictions of $|\ |_j$, $j = 1, \ldots, r$, to L are all extensions of $|\ |_v$ to absolute values on L and they are pairwise inequivalent.*

Proof: Suppose the restrictions to L of $|\ |_j$ and $|\ |_k$ are equivalent. Then there is an isomorphism $\varphi : K_j \xrightarrow{\sim} K_k$ which is the identity on L and K_v. Let φ_j be the canonical isomorphism of $L \otimes_K K_v$ onto K_j. Then $\varphi = \varphi_k \circ \varphi_j^{-1}$ (check on L and K_v), which is possible only if $j = k$. This proves the last clause of our claim.

Let $|\ |_w$ be an extension of $|\ |_v$ to L. The closure of K in L_w will be identified with K_v. Since L is finite dimensional over K, it follows that LK_v is a closed subfield of L_w, from which we conclude that $LK_v = L_w$. By the universal property of the tensor product, there is a homomorphism of $L \otimes_K K_v$ onto L_w. The kernel of this homomorphism is a maximal ideal, hence equal to $\mathfrak{m}_j \times \prod_{k \neq j} R_k$ for some j. Therefore, K_j is isomorphic to

L_w as a K_v-algebra, and by Proposition 1.2.7 we infer that it is in fact an isometry. This shows that the restriction of $|\ |_j$ to L is $|\ |_w$, as we wanted. $\square$

1.3.12. Now we are ready to define the correct normalizations of the absolute values as above. We have seen that the places w of L with $w|v$ are in one-to-one correspondence with the local K_v-algebras in (1.1), thus we may write

$$L \otimes_K K_v = \prod_{w|v} T_w \tag{1.2}$$

and identify the residue field of T_w with the completion L_w. For $y \in L$, we set

$$\|y\|_w := |N_{L_w/K_v}(y)|_v^{[T_w:L_w]}$$

and

$$|y|_w := \|y\|_w^{1/[L:K]}.$$

With these modifications the analogue of Lemma 1.3.7 still holds, namely:

Lemma 1.3.13. *If $x \in K \setminus \{0\}$ and $y \in L \setminus \{0\}$, then*

$$\sum_{w|v} \log |x|_w = \log |x|_v\,, \tag{1.3}$$

$$\sum_{w|v} \log \|y\|_w = \log |N_{L/K}(y)|_v\,. \tag{1.4}$$

Proof: Formula (1.4) is a trivial consequence of Proposition 1.2.7 and (1.2). If we set $y = x$ in (1.4), then (1.3) follows immediately from (1.2). $\square$

1.4. The product formula

The product formula over $\mathbb{Q}$ may be stated and proved as a consequence of the factorization of a non-zero rational number into a product of primes and a unit. In spite of its simplicity and essentially trivial nature, it plays a fundamental role and its importance cannot be overstated. The fact that it involves all places, including the places at ∞, means that, from the geometrical point of view, we are dealing with a complete variety. In the case considered here, the general fibre of the variety is a point and everything is quite simple. However, the best interpretation of the product formula and its generalizations is found in the framework of Arakelov theory.

1.4.1. Let K be a field and M_K be a set of non-trivial inequivalent absolute values on K such that the set

$$\{|\ |_v \in M_K \mid |x|_v \neq 1\}$$

is finite for any $x \in K\backslash\{0\}$. We identify the elements of M_K with the corresponding places and say that M_K satisfies the **product formula** if

$$\prod_{v \in M_K} |x|_v = 1$$

for any $x \in K\backslash\{0\}$.

We shall also refer to

$$\sum_{v \in M_K} \log |x|_v = 0$$

as the product formula for $x \neq 0$.

If L/K is a finite-dimensional extension and M_K is a set of places with associated normalized absolute values satisfying the product formula, we obtain a set of places M_L consisting of representatives $|\ |_w$ of $w|v$ for $v \in M_K$, normalized as in 1.3.6 and 1.3.12.

Proposition 1.4.2. *The set of places M_L so normalized again satisfies the product formula.*

Proof: Let $x \in L^\times$. We need to check that $|x|_w \neq 1$ only for finitely many $w \in M_L$. Since x is algebraic over K, we have

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{1.5}$$

for suitable $a_i \in K$. By assumption, we have $|a_i|_v \in \{0, 1\}$ up to finitely many $v \in M_K$. Since there are only finitely many $w \in M_L$ lying over a given v (Corollary 1.3.2), we have $|a_i|_w \leq 1$, up to finitely many $w \in M_L$. Clearly, there are only finitely many archimedean places in M_K and hence in M_L. Thus it is enough to consider non-archimedean $w \in M_L$ and then the ultrametric inequality applied to (1.5) shows that $|x|_w \leq 1$ whenever all coefficients $|a_i|_w \leq 1$. The same argument applied to $1/x$ completes the proof that $|x|_w = 1$ up to finitely many $w \in M_L$.

Once this is done, it is immediate from Lemmas 1.3.7 and 1.3.13 that the normalized set of absolute values on L satisfies the product formula. □

1.4.3. By Example 1.2.5, we get

$$M_{\mathbb{Q}} := \{|\ |_p \mid p \text{ prime number or } p = \infty\},$$

normalized as follows. If $p = \infty$, then $|\ |_p$ is the ordinary absolute value on $\mathbb{Q}$, and, if p is prime, then the absolute value is the p-adic absolute value on $\mathbb{Q}$, with $|p|_p = 1/p$.

Let K be a number field and let M_K be the associated set of places and normalized absolute values, obtained from the above construction applied to the extension $K/\mathbb{Q}$.

Proposition 1.4.4. *If K is a number field, M_K satisfies the product formula.*

Proof: By the above discussion, we can assume that K is equal to $\mathbb{Q}$ and it is obviously enough to show the product formula for a prime number x

$$\prod_{p \in M_{\mathbb{Q}}} |x|_p = |x|_x\ |x|_\infty = \frac{1}{x}\, x = 1. \qquad \square$$