Introduction to General Relativity

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For Mildred Elizabeth Ryder

It is always a source of pleasure when a great and beautiful idea proves to be correct in actual fact. Albert Einstein [letter to Sigmund Freud]

The answer to all these questions may not be simple. I know there are some scientists who go about preaching that Nature always takes on the simplest solutions. Yet the simplest by far would be nothing, that there would be nothing at all in the universe. Nature is far more interesting than that, so I refuse to go along thinking it always has to be simple.
Richard Feynman

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Preface

This book is designed for final year undergraduates or beginning graduate students in physics or theoretical physics. It assumes an acquaintance with Special Relativity and electromagnetism, but beyond that my aim has been to provide a pedagogical introduction to General Relativity, a subject which is now – at last – part of mainstream physics. The coverage is fairly conventional; after outlining the need for a theory of gravity to replace Newton's, there are two chapters devoted to differential geometry, including its modern formulation in terms of differential forms and coordinate-free vectors, then the Einstein field equations, the Schwarzschild solution, the Lense–Thirring effect (recently confirmed observationally), black holes, the Kerr solution, gravitational radiation and cosmology. The book ends with a chapter on field theory, describing similarities between General Relativity and gauge theories of particle physics, the Dirac equation in Riemannian space-time, and Kaluza–Klein theory.

As a research student I was lucky enough to attend the Les Houches summer school in 1963 and there, in the magnificent surroundings of the French alps, began an acquaintance with many of the then new aspects of this subject, just as it was entering the domain of physics proper, eight years after Einstein's death. A notable feature was John Wheeler's course on gravitational collapse, before he had coined the phrase 'black hole'. In part I like to think of this book as passing on to the community of young physicists, after a gap of more than 40 years, some of the excitement generated at that school.

I am very grateful to the staff at Cambridge University Press, Tamsin van Essen, Lindsay Barnes and particularly Simon Capelin for their unfailing help and guidance, and generosity over my failure to meet deadlines. I also gratefully acknowledge helpful conversations and correspondence with Robin Tucker, Bahram Mashhoon, Alexander Shannon, the late Jeeva Anandan, Brian Steadman, Daniel Ryder and especially Andy Hone, who have all helped to improve my understanding. Finally I particularly want to thank my wife, who has supported me throughout this long project, with constant good humour and generous and selfless encouragement. To her the book is dedicated.

Notation, important formulae and physical constants

Latin indices i, j, k, and so on run over the three spatial coordinates 1, 2, 3 or x, y, z or r, θ, φ
Greek indices $\alpha, \beta, \gamma, \ldots \kappa, \lambda, \mu, \ldots$ and so on run over the four space-time coordinates 0, 1, 2, 3 or ct, x, y, z or ct, r, θ, ϕ
Minkowski space-time: metric tensor is $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1)$, $\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ in Cartesian coordinates
Riemannian space-time: $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\,c^2\,\mathrm{d}\tau^2$
The Levi-Cività totally antisymmetric symbol (in Minkowski space) is

$$\varepsilon^{0123} = -\varepsilon_{0123} = 1$$

Connection coefficients: $\Gamma^\nu_{\mu\kappa} = {}^1\!/_2\, g^{\nu\rho}(g_{\mu\rho,\kappa} + g_{\kappa\rho,\mu} - g_{\mu\kappa,\rho})$
Riemann tensor: $R^\kappa{}_{\lambda\mu\nu} = \Gamma^\kappa{}_{\lambda\nu,\mu} - \Gamma^\kappa{}_{\lambda\mu,\nu} + \Gamma^\kappa{}_{\rho\mu}\Gamma^\rho{}_{\lambda\nu} - \Gamma^\kappa{}_{\rho\nu}\Gamma^\rho{}_{\lambda\mu}$
Ricci tensor: $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$
Curvature scalar: $R = g^{\mu\nu}R_{\mu\nu}$
Field equations: $G_{\mu\nu} = R_{\mu\nu} - {}^1\!/_2\, g_{\mu\nu}R = \dfrac{8\pi G}{c^2}T_{\mu\nu}$
Covariant derivatives:

$$\frac{\mathrm{D}V^\mu}{\mathrm{d}x^\nu} = \frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\lambda\nu}V^\lambda \quad \text{or} \quad V^\mu{}_{;\nu} = V^\mu{}_{,\nu} + \Gamma^\mu{}_{\lambda\nu}V^\lambda$$

$$\frac{\mathrm{D}W_\mu}{\mathrm{d}x^\nu} = \frac{\partial W_\mu}{\partial x^\nu} - \Gamma^\lambda{}_{\mu\nu}W_\lambda \quad \text{or} \quad W_{\mu;\nu} = W_{\mu,\nu} - \Gamma^\lambda{}_{\mu\nu}W_\lambda$$

Speed of light	$c = 3.00 \times 10^8\ \mathrm{m\,s^{-1}}$
Gravitational constant	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-1}}$
Planck's constant	$\hbar = 1.05 \times 10^{-34}\ \mathrm{J\,s}$
	$= 6.58 \times 10^{-22}\ \mathrm{MeV\,s}$
Electron mass	$m_\mathrm{e} = 9.11 \times 10^{-31}\ \mathrm{kg}$
	$m_\mathrm{e}c^2 = 0.51\ \mathrm{MeV}$
Proton mass	$m_\mathrm{p} = 1.672 \times 10^{-27}\ \mathrm{kg}$
	$m_\mathrm{p}c^2 = 938.3\ \mathrm{MeV}$
Neutron mass	$m_\mathrm{n} = 1.675 \times 10^{-27}\ \mathrm{kg}$
	$m_\mathrm{n}c^2 = 939.6\ \mathrm{MeV}$
Boltzmann constant	$k = 1.4 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
	$= 8.6 \times 10^{-11}\ \mathrm{MeV\,K^{-1}}$
Solar mass	$M_\mathrm{S} = 1.99 \times 10^{30}\ \mathrm{kg}$

Solar radius	$R_S = 6.96 \times 10^8$ m
Earth mass	$M_E = 5.98 \times 10^{24}$ kg
Earth equatorial radius	$R_E = 6.38 \times 10^6$ m
Mean Earth–Sun distance	$R = 1.50 \times 10^{11}$ m = 1 AU
Schwarzschild radius of Sun	$2m = \dfrac{2M_S G}{c^2} = 2.96$ km
Stefan–Boltzmann constant	$\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}

1 light year (ly) = 9.46×10^{15} m
1 pc = 3.09×10^{16} m = 3.26 ly
1 radian = 2.06×10^5 seconds of arc