1 Introduction

Einstein's General Theory of Relativity, proposed in 1916, is a theory of gravity. It is also, as its name suggests, a generalisation of Special Relativity, which had been proposed in 1905. This immediately suggests two questions. Firstly, why was a new theory of gravity needed? Newton's theory was, to put it mildly, perfectly good enough. Secondly, why is it that a generalisation of Special Relativity yields, of all things, a theory of *gravity*? Why doesn't it give a theory of electromagnetism, or the strong or weak nuclear forces? Or something even more exotic? What is so special about gravity, that generalising a theory of space and time (because that is what Special Relativity is) gives us an account of it? We begin this chapter by answering the first question first. By the end of the chapter we shall also have made a little bit of headway in the direction of answering the second one.

1.1 The need for a theory of gravity

Newton's theory of gravitation is a spectacularly successful theory. For centuries it has been used by astronomers to calculate the motions of the planets, with a staggering success rate. It has, however, the fatal flaw that it is inconsistent with Special Relativity. We begin by showing this.

As every reader of this book knows, Newton's law of gravitation states that the force exerted on a mass m by a mass M is

$$\mathbf{F} = -\frac{MmG}{r^3}\mathbf{r}. \tag{1.1}$$

Here M and m are not necessarily point masses; r is the distance between their centres of mass. The vector $\mathbf{r}$ has a direction from M to m. Now suppose that the mass M depends on time. The above formula will then become

$$\mathbf{F}(t) = -\frac{M(t)mG}{r^3}\mathbf{r}. \tag{1.2}$$

This means that the force felt by the mass m at a time t depends on the value of the mass M *at the same time* t. There is no allowance for time delay, as Special Relativity would require. From our experience of advanced and retarded potentials in electrodynamics, we can say that Special Relativity would be satisfied if, in the above equation, $M(t)$ were modified to $M(t - r/c)$. This would reflect the fact that the force felt by the small mass at time t depended on the value of the large mass at an *earlier* time $t - r/c$; assuming, that is, that the relevant gravitational 'information' travelled at the speed of light. But this would then not be Newton's law. Newton's law is Equation (1.2) which allows for no time delay, and therefore implicitly suggests that the *information* that the mass M is changing travels with infinite velocity, since the effect of a changing M is felt at the same instant by the mass m. Since Special Relativity implies that nothing can travel faster than light, Equations (1.1) and (1.2) are incompatible with it. If two theories are incompatible, at least one of them must be wrong. The only possible attitude to adopt is that Special Relativity must be kept intact, so Newton's law has to be changed.

Faced with such a dramatic situation – not to say crisis – the instinctive, and perfectly sensible, reaction of most physicists would be to try to 'tinker' with Newton's law; to change it slightly, in order to make it compatible with Special Relativity. And indeed many such attempts were made, but none were successful.[1] Einstein eventually concluded that nothing less than a complete 'new look' at the problem of gravitation had to be taken. We shall return to this in the next section, but before leaving this one it will be useful to rewrite the above equations in a slightly different form; it should be clear that, although Newton's equations are 'wrong', they are an extremely good approximation to whatever 'correct' theory is eventually found, so this theory should then give, as a first approximation, Newton's law. We have by no means finished with Newton!

Let us define $\mathbf{g} = \mathbf{F}/m$, the gravitational *field intensity*. This is a parallel equation to $\mathbf{E} = \mathbf{F}/q$ in electrostatics; the electric field is the force per unit charge and the gravitational field the force per unit mass. Mass is the 'source' of the gravitational field in the same way that electric charge is the source of an electric field. Then Equation (1.1) can be written

$$\mathbf{g} = -\frac{GM}{r^2}\hat{r}, \tag{1.3}$$

which gives an expression for the gravitational field intensity at a distance r from a mass M. This expression, however, is of a rather special form, since the right hand side is a *gradient*. We can write

$$\mathbf{g} = -\nabla\phi, \quad \phi(r) = -\frac{GM}{r}. \tag{1.4}$$

The function $\phi(r)$ is the gravitational *potential*, a scalar field. Newton's theory is then described simply by *one function*. (In contrast, as we shall see in due course, the gravitational field in General Relativity is described by *ten* functions, the ten components of the metric tensor. The non-relativistic limit of *one* of these components is, in essence, the Newtonian potential.) A mass, or a distribution of masses, gives rise to a scalar gravitational potential that completely determines the gravitational field. The potential ϕ in turn satisfies *field equations*. These are Laplace's and Poisson's equations, relevant, respectively, to the cases where there is a vacuum, or a matter density ρ:

$$\text{(Laplace)} \qquad \nabla^2\phi = 0 \qquad \text{(vacuum)}, \tag{1.5}$$

$$\text{(Poisson)} \qquad \nabla^2\phi = 4\pi G\rho \qquad \text{(matter)}. \tag{1.6}$$

[1] For references to these see 'Further reading' at the end of the chapter.

In the case of a point mass, of course, we have $\rho\,(r) = M\,\delta^3(\mathbf{r})$, and by virtue of the identity

$$\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r}) \tag{1.7}$$

Equations (1.4) and (1.6) are in accord.

This completes our account of Newtonian gravitational theory. The field **g** depends on r but not on t. Such a field is incompatible with Special Relativity. It is not a Lorentz covariant field; such a field would be a *four*-vector rather than a three-vector and would depend on t as well as on r, so that the equations of gravity looked the same in all frames of reference related by Lorentz transformations. This is not the case here. Since Newton's theory is inconsistent with Special Relativity it must be abandoned. This is both a horrifying prospect and a slightly encouraging one; horrifying because we are having to abandon one of the best theories in physics, and encouraging because Newton's theory is so precise and so successful that any new theory of gravity will immediately have to fulfil the very stringent requirement that in the non-relativistic limit it should yield Newton's theory. This will provide an immediate test for a new theory.

1.2 Gravitation and inertia: the Equivalence Principle in mechanics

Einstein's new approach to gravity sprang from the work of Galileo (1564–1642; he was born in the same year as Shakespeare and died the year Newton was born). Galileo conducted a series of experiments rolling spheres down ramps. He varied the angle of inclination of the ramp and timed the spheres with a water clock. Physicists commonly portray Galileo as dropping masses from the Leaning Tower of Pisa and timing their descent to the ground. Historians cast doubt on whether this happened, but for our purposes it hardly matters whether it did or didn't; what matters is the conclusion Galileo drew. By extrapolating to the limit in which the ramps down which the spheres rolled became vertical, and therefore that the spheres fell freely, he concluded that *all bodies fall at the same rate in a gravitational field*. This, for Einstein, was a crucially important finding. To investigate it further consider the following 'thought-experiment', which I refer to as 'Einstein's box'. A box is placed in a gravitational field, say on the Earth's surface (Fig. 1.1(a)). An experimenter in the box releases two objects, made of different materials, from the same height, and measures the times of their fall in the gravitational field **g**. He finds, as Galileo found, that they reach the floor of the box at the same time. Now consider the box in free space, completely out of the reach of any gravitational influences of planets or stars, but subject to an *acceleration* **a** (Fig. 1.1(b)). Suppose an experimenter in this box also releases two objects at the same time and measures the time which elapses before they reach the floor. He will find, of course, that they take the same time to reach the floor; he *must* find this, because when the two objects are released, they are then subject to no force, because no acceleration, and it is the floor of the box that accelerates up to meet them. It clearly reaches them at the same time. We conclude that this experimenter, by releasing objects and timing their fall, *will not be able to tell whether he is in a gravitational field or being accelerated through*

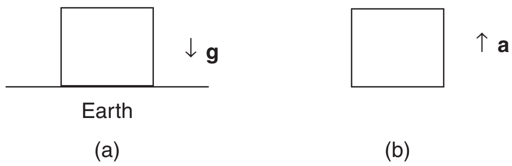

Fig. 1.1 The Einstein box: a comparison between a gravitational field and an accelerating frame of reference.

empty space. The experiments will give identical results. A gravitational field is therefore *equivalent* to an accelerating frame of reference – at least, as measured in this experiment. This, according to Einstein, is the significance of Galileo's experiments, and it is known as the *Equivalence Principle*. Stated in a more general way, the Equivalence Principle says that *no experiment in mechanics can distinguish between a gravitational field and an accelerating frame of reference*. This formulation, the reader will note, already goes beyond Galileo's experiments; the claim is made that *all* experiments in mechanics will yield the same results in an accelerating frame and in a gravitational field. Let us now analyse the consequences of this.

We begin by considering a particle subject to an acceleration **a**. According to Newton's second law of motion, in order to make a particle accelerate it is necessary to apply a force to it. We write

$$\mathbf{F} = m_{\mathrm{i}}\mathbf{a}. \tag{1.8}$$

Here m_{i} is the *inertial mass* of the particle. The above law states that the reason a particle needs a force to accelerate it is that the particles possesses *inertia*. A very closely related idea is that acceleration is *absolute*; (constant) velocity, on the other hand, is *relative*. Now consider a particle falling in a gravitational field **g**. It will experience a force (see (1.2) and (1.3) above) given by

$$\mathbf{F} = m_{\mathrm{g}}\mathbf{g}. \tag{1.9}$$

Here m_{g} is the *gravitational mass* of the particle. It measures the response of a particle to a *gravitational field*. It is very important to appreciate that gravitational mass and inertial mass are conceptually *entirely distinct*. Acceleration in free space is an entirely different thing from a gravitational field, and we make this distinction clear by distinguishing gravitational and inertial mass, as in the two equations above. Now, however, consider a particle falling freely in a gravitational field, as in the Einstein box experiments. Both equations above apply. Because the particle is in a gravitational field it will experience a force, given by (1.9); and because a force is acting on the particle it will accelerate, the acceleration being given by (1.8). These two equations then give

$$\mathbf{a} = \frac{\mathbf{F}}{m_{\mathrm{i}}} = \frac{m_{\mathrm{g}}}{m_{\mathrm{i}}}\mathbf{g}; \tag{1.10}$$

the acceleration of a particle in a gravitational field **g** is the ratio of its gravitational and inertial masses times **g**. Galileo's experiments therefore imply that $m_{\mathrm{g}}/m_{\mathrm{i}}$ *is the same for all* *materials*. Without loss of generality we may put $m_g = m_i$ for all materials; this is because the formula for **g** contains G (see (1.3)), so by scaling G, m_g/m_i can be made equal to unity. (In fact, of course, historically G was found by *assuming* that $m_g = m_i$; no distinction was made between gravitational and inertial masses. We are now 'undoing' history.) We conclude that the Equivalence Principle states that

$$m_g = m_i. \tag{1.11}$$

Gravitational mass is the *same as* inertial mass for *all materials*. This is an interesting and non-trivial result. Some very sensitive experiments have been performed, and continue to be performed, to test this equality to higher and higher standards of accuracy. After Galileo, the most interesting experiment was done by Eötvös and will be described below. Before that, however, it is worth devoting a few minutes' thought to the significance of the equality (1.11) above.

The inertial mass of a piece of matter has contributions from two sources; the mass of the 'constituents' and the binding energy, expressed in mass units ($m = E/c^2$). This is the case no matter what the type of binding. So for example the mass of an atom is the sum of the masses of its constituent protons and neutrons *minus the nuclear binding energy* (divided by c^2). In the case of nuclei, the binding energy makes a contribution of the order of 10^{-3} to the total mass. Atoms are bound together by electromagnetic forces and stars and planets are bound by gravitational forces. In all of these cases, the binding energy, as well as the inertial mass of the constituents, contributes to the overall inertial mass of the sample. The statement (1.11) above then implies that the binding energy of a body will *also* contribute to its *gravitational* mass, so binding energy (in fact, energy in general) has a *gravitational effect* since its mass equivalent will in turn give rise to a gravitational field. The gravitational force itself, by virtue of the binding it gives rise to, also gives rise to further gravitational effects. In this sense gravity is *non-linear*. Electromagnetism, on the other hand, is linear; electromagnetic forces give rise to (binding) energy, which acts as a source of gravity, but *not* as a source of further electromagnetic fields, since electromagnetic energy possesses *no charge*. Gravitational energy, however, possesses an effective mass and therefore gives rise to further gravitational fields.

Now let us turn to experiments to test the Equivalence Principle. The simplest one to imagine is simply the measurement of the displacement from the vertical with which a large mass hangs, in the gravitational field of the (rotating) Earth. From Problem 1.1 we see that this displacement is (in Budapest) of the order of 6 minutes of arc multiplied by m_g/m_i. To see whether m_g/m_i is the same for all substances, then, involves looking for tiny variations in this angle, for masses made of differing materials. This is a very difficult measurement to make, not least because it is *static*.

A better test for the constancy of m_g/m_i relies on the gravitational attraction of the *Sun*, whose position relative to the Earth varies with a 24 hour period. We are therefore looking for a periodic signal, which stands more chance of being observed above the noise than does a static one. The simplest version of this is the *Eötvös* or *torsion balance*; the original torsion balance was invented by Coulomb and by Mitchell, and was used by Cavendish to verify the inverse square law of gravity. For the purposes of this experiment the torsion balance takes the form shown in Fig. 1.2.

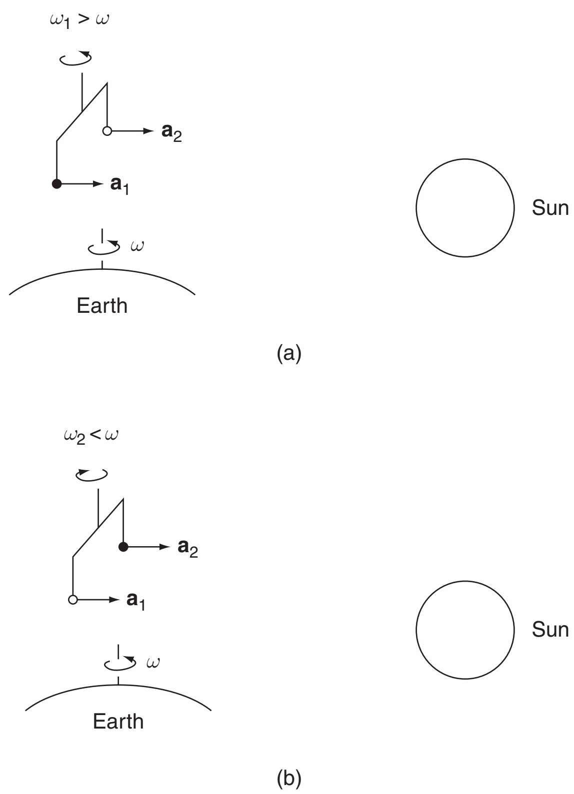

Fig. 1.2 A torsion balance at the North Pole. (a) and (b) represent two situations with a 12 hour time separation. The Earth is rotating with angular velocity ω and $\mathbf{a}_1$ and $\mathbf{a}_2$ are the accelerations of the gold and aluminium masses towards the Sun. Assuming that $\mathbf{a}_1 > \mathbf{a}_2$ the resulting torques are of opposite sign.

Two masses, one of gold (shaded) one of aluminium (not shaded), hang from opposite ends of an arm suspended by a thread in the gravitational field of the Earth. Consider such a balance at the North Pole, with the Sun in some assigned position to the right of the diagram. Then at 6 a.m., say, the situation is as shown in (a), the Earth rotating with angular velocity ω. The force exerted by the Sun on the gold mass is (M is the mass of the Sun and r the Earth–Sun distance)

$$F_{\mathrm{Au}} = \frac{GM(m_{\mathrm{g}})_{\mathrm{Au}}}{r^2} \tag{1.12}$$

and hence its acceleration towards the Sun is

$$a_{\mathrm{Au}} = \frac{GM}{r^2}\left(\frac{m_{\mathrm{g}}}{m_{\mathrm{i}}}\right)_{\mathrm{Au}}. \tag{1.13}$$

A similar formula holds for the aluminium mass. Putting

$$\frac{m_{\mathrm{g}}}{m_{\mathrm{i}}} = 1 + \delta, \tag{1.14}$$

then if $\delta_{\rm Au} \neq \delta_{\rm Al}$ a *torque* is exerted on the balance, of magnitude ($2l$ is the length of the arm)

$$T = \frac{GMl}{r^2}[(m_{\rm g})_{\rm Au} - (m_{\rm g})_{\rm Al}]. \tag{1.15}$$

This results in an angular acceleration α given by $T = I\alpha$, with I, the moment of inertia, given by $I = m_{\rm i}l^2$, so we have, at 6 a.m.,

$$(\alpha)_{\rm 6am} = \frac{GM}{lr^2}(\delta_{\rm Au} - \delta_{\rm Al}) \equiv \frac{GM}{lr^2}\Delta, \tag{1.16}$$

where $\Delta = \delta_{\rm Au} - \delta_{\rm Al}$. In diagram (a) we suppose that $\Delta > 0$, i.e. the acceleration of the gold mass is greater than that of the aluminium mass. This in effect causes the torsion balance to rotate with angular velocity $\omega_1 > \omega$. At 6 p.m., however, the situation is reversed (Fig. 1.2(b)) so the direction of the torque will be reversed, and

$$(\alpha)_{\rm 6pm} = -\frac{GM}{lr^2}\Delta. \tag{1.17}$$

Thus there would be a periodic variation in the torque, with a period of 24 hours. No such variation has been observed,[2] allowing the conclusion that

$$\delta < 10^{-11}; \tag{1.18}$$

gravitational mass and inertial mass are equal to one part in 10^{11} – at least as measured using gold and aluminium.

1.1.1 A remark on inertial mass

The Equivalence Principle states the equality of gravitational and inertial mass, as we have just seen above. It is worthwhile, however, making the following remark. The inertial mass of a particle refers to its mass (deduced, for example, from its behaviour analysed according to Newton's laws) when it undergoes non-uniform, or non-inertial, motion. There are, however, two different types of such motion; it may for instance be acceleration in a straight line, or circular motion with constant speed. In the first case the magnitude of the velocity vector changes but its direction remains constant, while in the second case the magnitude is constant but the direction changes. In each of these cases the motion is non-inertial, but there is a conceptual distinction to be made. To be precise we should observe this distinction and denote the two types of mass $m_{\rm i,acc}$ and $m_{\rm i,rot}$. We believe, without, as far as I know, proper evidence, that they are equal

$$m_{\rm i,acc} = m_{\rm i,rot}. \tag{1.19}$$

The interesting thing is that Einstein's formulation of the Equivalence Principle referred to inertial mass measured in an accelerating frame, $m_{\rm i,acc}$, whereas the Eötvös experiment, described above, establishes the equality (to within the stated bounds) of $m_{\rm i,rot}$ and the

[2] Roll *et al.* (1964), Braginsky & Panov (1972).

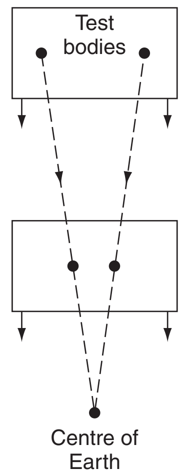

Fig. 1.3 **Test bodies falling to the centre of the Earth.**

gravitational mass. The question is: can an experiment be devised to test the equality of $m_{\mathrm{i,acc}}$ and m_{g}? Or even to test (1.19)?

1.1.2 Tidal forces

The Principle of Equivalence is a *local* principle. To see this, consider the Einstein box in the gravitational field of the Earth, as in Fig. 1.3. If the box descends over a large distance towards the centre of the Earth, it is clear that two test bodies in the box will approach one another, so over this *extended* journey it is clear that they are in a genuine gravitational field, and *not* in an accelerating frame (in which they would stay the same distance apart). In other words, the Equivalence Principle has broken down. We conclude that this principle is only valid as a *local* principle. Over small distances a gravitational field is equivalent to an acceleration, but over larger distances this equivalence breaks down. The effect is known as a *tidal effect*, and ultimately is due to the *curvature* produced by a real gravitational field.

Another way of stating the situation is to note that an object in free fall is in an inertial frame. The effect of the gravitational field has been *cancelled* by the acceleration of the elevator (the ‘acceleration due to gravity’). The accelerations required to annul the gravitational fields of the two test bodies, however, are slightly different, because they are directed along the radius vectors. So the inertial frames of the two bodies differ slightly. The frames are ‘locally inertial’. The Equivalence Principle treats a gravitational field *at a single point* as equivalent to an acceleration, but it is clear that no gravitational fields encountered in nature give rise to a *uniform* acceleration. Most real gravitational fields are produced by more or less spherical objects like the Earth, so the equivalence in question is only a local one.

We may find an expression for the tidal forces which result from this non-locality. Figure 1.4 shows the forces exerted on the two test bodies – call them A and B – in the gravitational field of a body at O. They both experience a force towards O of magnitude

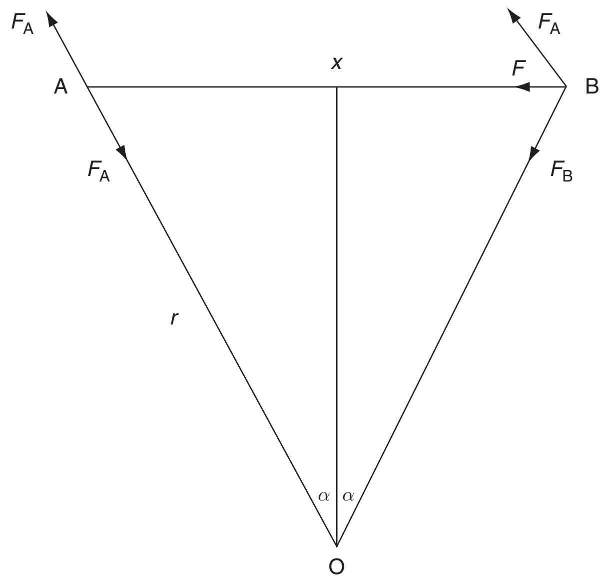

Fig. 1.4 Tidal effect: forces on test bodies A and B.

$$F_A = F_B = \frac{mMG}{r^2}$$

where m is the mass of A and B, M is the mass of the Earth and r the distance of A and B from its centre. In addition, let the distance between A and B be x. Consider the frame in which A is at rest. This frame is realised by applying a force equal and opposite to F_A, to both A and B, as shown in Fig. 1.4. In this frame, B experiences a force F, directed towards A, which is the vector sum of F_B and $-F_A$:

$$F = 2F_A \sin\alpha = 2F_A \cdot \frac{x}{2r} = \frac{mMG}{r^3}x.$$

A then observes B to be accelerating towards him with an acceleration given by $F = -m\,d^2x/dt^2$, i.e.

$$\frac{d^2x}{dt^2} = -\frac{MG}{r^3}x. \tag{1.20}$$

The $1/r^3$ behaviour is characteristic of tidal forces.

1.3 The Equivalence Principle and optics

The Equivalence Principle is a principle of *indistinguishability*; it is impossible, using any experiment in mechanics, to *distinguish* between a gravitational field and an accelerating frame of reference. To this extent it is a *symmetry principle*. If a symmetry of nature is *exact*,

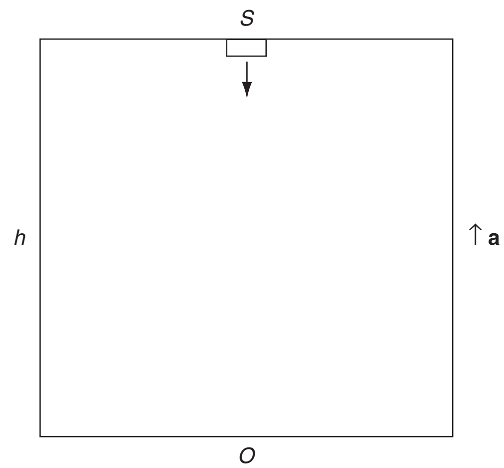

Fig. 1.5 **Light propagating downwards in a box accelerating upwards.**

this means that various situations are experimentally indistinguishable. If, for example, parity were an exact symmetry of the world (which it is not, because of beta decay), it would be impossible to distinguish left from right. The fact that it *is* possible to distinguish them is a direct indication of the breaking of the symmetry.

No experiment in mechanics, then, can distinguish a gravitational field from an accelerating frame. What about other areas of physics? Let us generalise the Equivalence Principle to *optics*, and consider the idea that no experiment in optics could distinguish a gravitational field from an accelerating frame.[3] To make this concrete, return to the Einstein box and consider the following simple two experiments. The first one is to release monochromatic light (of frequency ν) from the ceiling of the accelerating box, and receive it on the floor (Fig. 1.5). The light is released from the source S at $t=0$ towards the observer O. At the same instant $t=0$ the box begins to accelerate upwards with acceleration a. The box is of height h. Light from S reaches O after a time interval $t=h/c$, at which time O is moving upwards with speed $u=at=ah/c$.

Now consider the emission of two successive crests of light from S. Let the time interval between the emission of these crests be $\mathrm{d}t$ in the frame of S. Then

$$\mathrm{d}t = \frac{1}{\nu} \quad \text{in frame } S, \tag{1.21}$$

where ν is the frequency of the light in frame S. Arguing *non-relativistically*, the time interval between the *reception* of these crests at O is

$$\mathrm{d}t' = \mathrm{d}t - \Delta t = \mathrm{d}t - u\frac{\mathrm{d}t}{c} = \mathrm{d}t\left(1 - \frac{u}{c}\right) = \frac{1}{\nu'},$$

[3] This generalisation is sometimes characterised as a progression from a Weak Equivalence Principle (which is the statement $m_\mathrm{i} = m_\mathrm{g}$) to a Strong Equivalence Principle, according to which all the laws of nature (not just those of freely falling bodies) are affected in the same way by a gravitational field and a constant acceleration.