Elements of Distribution Theory

This detailed introduction to distribution theory uses no measure theory, making it suitable for students in statistics and econometrics as well as for researchers who use statistical methods. Good backgrounds in calculus and linear algebra are important and a course in elementary mathematical analysis is useful, but not required. An appendix gives a detailed summary of the mathematical definitions and results that are used in the book.

Topics covered range from the basic distribution and density functions, expectation, conditioning, characteristic functions, cumulants, convergence in distribution, and the central limit theorem to more advanced concepts such as exchangeability, models with a group structure, asymptotic approximations to integrals, orthogonal polynomials, and saddlepoint approximations. The emphasis is on topics useful in understanding statistical methodology; thus, parametric statistical models and the distribution theory associated with the normal distribution are covered comprehensively.

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To My Parents

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Preface

Distribution theory lies at the interface of probability and statistics. It is closely related to probability theory; however, it differs in its focus on the calculation and approximation of probability distributions and associated quantities such as moments and cumulants. Although distribution theory plays a central role in the development of statistical methodology, distribution theory itself does not deal with issues of statistical inference.

Many standard texts on mathematical statistics and statistical inference contain either a few chapters or an appendix on basic distribution theory. I have found that such treatments are generally too brief, often ignoring such important concepts as characteristic functions or cumulants. On the other hand, the discussion in books on probability theory is often too abstract for readers whose primary interest is in statistical methodology.

The purpose of this book is to provide a detailed introduction to the central results of distribution theory, in particular, those results needed to understand statistical methodology, without requiring an extensive background in mathematics. Chapters 1 to 4 cover basic topics such as random variables, distribution and density functions, expectation, conditioning, characteristic functions, moments, and cumulants. Chapter 5 covers parametric families of distributions, including exponential families, hierarchical models, and models with a group structure. Chapter 6 contains an introduction to stochastic processes.

Chapter 7 covers distribution theory for functions of random variables and Chapter 8 covers distribution theory associated with the normal distribution. Chapters 9 and 10 are more specialized, covering asymptotic approximations to integrals and orthogonal polynomials, respectively. Although these are classical topics in mathematics, they are often overlooked in statistics texts, despite the fact that the results are often used in statistics. For instance, Watson's lemma and Laplace's method are general, useful tools for approximating the integrals that arise in statistics, and orthogonal polynomials are used in areas ranging from nonparametric function estimation to experimental design.

Chapters 11 to 14 cover large-sample approximations to probability distributions. Chapter 11 covers the basic ideas of convergence in distribution and Chapter 12 contains several versions of the central limit theorem. Chapter 13 considers the problem of approximating the distribution of statistics that are more general than sample means, such as nonlinear functions of sample means and *U*-statistics. Higher-order asymptotic approximations such as Edgeworth series approximations and saddlepoint approximations are presented in Chapter 14.

I have attempted to keep each chapter as self-contained as possible, but some dependencies are inevitable. Chapter 1 and Sections 2.1–2.4, 3.1–3.2, and 4.1-4.4 contain core topics that are used throughout the book; the material covered in these sections will most likely be familiar to readers who have taken a course in basic probability theory. Chapter 12 requires Chapter 11 and Chapters 13 and 14 require Chapter 12; in addition, Sections 13.3 and 13.5 use material from Sections 7.5 and 7.6.

The mathematical prerequisites for this book are modest. Good backgrounds in calculus and linear algebra are important and a course in elementary mathematical analysis at the level of Rudin (1976) is useful, but not required. Appendix 3 gives a detailed summary of the mathematical definitions and results that are used in the book.

Although many results from elementary probability theory are presented in Chapters 1 to 4, it is assumed that readers have had some previous exposure to basic probability theory. Measure theory, however, is not needed and is not used in the book. Thus, although measurability is briefly discussed in Chapter 1, throughout the book all subsets of a given sample space are implictly assumed to be measurable. The main drawback of this is that it is not possible to rigorously define an integral with respect to a distribution function and to establish commonly used properties of this integral. Although, ideally, readers will have had previous exposure to integration theory, it is possible to use these results without fully understanding their proofs; to help in this regard, Appendix 1 contains a brief summary of the integration theory needed, along with important properties of the integral.

Proofs are given for nearly every result stated. The main exceptions are results requiring measure theory, although there are surprisingly few results of this type. In these cases, I have tried to outline the basic ideas of the proof and to give an indication of why more sophisticated mathematical results are needed. The other exceptions are a few cases in which a proof is given for the case of real-valued random variables and the extension to random vectors is omitted and a number of cases in which the proof is left as an exercise. I have not attempted to state results under the weakest possible conditions; on the contrary, I have often imposed relatively strong conditions if that allows a simpler and more transparent proof.

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