MULTISCALE STOCHASTIC VOLATILITY FOR EQUITY, INTEREST RATE, AND CREDIT DERIVATIVES

Building upon the ideas introduced in their previous book, *Derivatives in Financial Markets with Stochastic Volatility*, the authors study the pricing and hedging of financial derivatives under stochastic volatility in equity, interest rate, and credit markets. They present and analyze multiscale stochastic volatility models and asymptotic approximations. These can be used in equity markets, for instance, to link the prices of path-dependent exotic instruments to market implied volatilities. The methods are also used for interest rate and credit derivatives. Other applications considered include variance-reduction techniques, portfolio optimization, forward-looking estimation of CAPM "beta," and the Heston model and generalizations of it.

"Off-the-shelf" formulas and calibration tools are provided to ease the transition for practitioners who adopt this new method. The attention to detail and explicit presentation make this also an excellent text for a graduate course in financial and applied mathematics.

JEAN-PIERRE FOUQUE studied at the University Pierre et Marie Curie in Paris. He held positions at the French CNRS and Ecole Polytechnique, and at North Carolina State University. Since 2006, he is Professor and Director of the Center for Research in Financial Mathematics and Statistics at the University of California Santa Barbara.

GEORGE PAPANICOLAOU was Professor of Mathematics at the Courant Institute before coming to Stanford University in 1993. He is now Robert Grimmett Professor in the Department of Mathematics at Stanford.

RONNIE SIRCAR is a Professor in the Operations Research and Financial Engineering department at Princeton University, and an affiliate member of the Bendheim Center for Finance and the Program in Applied and Computational Mathematics.

KNUT SØLNA is a Professor in the Department of Mathematics at the University of California at Irvine. He received his undergraduate and Master's degrees from the Norwegian University of Science and Technology, and his doctorate from Stanford University. He was an instructor at the Department of Mathematics, University of Utah before coming to Irvine.

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JEAN-PIERRE FOUQUE
University of California, Santa Barbara

GEORGE PAPANICOLAOU
Stanford University

RONNIE SIRCAR
Princeton University

KNUT SØLNA
University of California, Irvine

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To our families and students

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Jean-Pierre Fouque , George Papanicolaou , Ronnie Sircar , Knut Sølna
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Introduction

This book is about pricing and hedging financial derivatives under stochastic volatility in equity, interest rate, and credit markets. We demonstrate that the introduction of two time scales in volatility, a fast and a slow, is needed and is efficient for capturing the main features of the observed term structures of implied volatility, yields, or credit spreads. The present book builds on and replaces our previous book, *Derivatives in Financial Markets with Stochastic Volatility*, published by Cambridge University Press in 2000.

We present an approach to derivatives valuation and hedging which consists of integrating singular and regular perturbation techniques in the context of stochastic volatility. The book has a dual purpose: to present "off-the-shelf" formulas and calibration tools, and to introduce, explain, and develop the mathematical framework to handle the multiscale asymptotics.

There are many books on financial mathematics (mostly for introductory courses at the level of the Black–Scholes model). Primarily, these books deal with the case of constant volatilities, be it for stock prices, interest rates, or default intensities. This book is about analyzing these models in the presence of stochastic volatility using the powerful tools of perturbation methods. The book can be used for a second-level graduate course in Financial and Applied Mathematics.

Our goal is to address the following fundamental problem in pricing and hedging derivatives: how can traded call and put options, quoted in terms of implied volatilities, be used to price and hedge more complicated contracts? Modeling the underlying asset usually involves the specification of a multifactor Markovian model under the risk-neutral pricing measure. Calibration of the parameters of that model to the observed implied volatilities, including the market prices of risk, is a challenging task because of the complex relation between option prices and model parameters (through a pricing partial differential equation, for instance). The main difficulty is to find models which will produce stable parameter estimates. We like to think of this problem as the "(K,T,t)-problem": for a given present time t and a fixed maturity T, it is usually easy with low-dimensional models to fit the skew with respect to strikes K. Getting a good fit of the term structure of implied volatility, that is when a range of observed maturities T is taken into account, is a much harder problem that can be handled with a sufficient number of parameters. The main problem remains: the stability with respect to t of these calibrated parameters. This is a crucial quality to have if one wants to use the model to compute no-arbitrage prices of more complex path-dependent derivatives, since in this case the distribution over time of the underlying is central.

Modeling directly the evolution of the implied volatility surface is a promising approach but involves some complicated issues. One has to make sure that the model is free of arbitrage or, in other words, that the surface is produced by some underlying under a risk-neutral measure. This is known to be a difficult task, and the choice of a model and its calibration is also an important issue in this approach. But most importantly, in order to use this modeling to price other path-dependent contracts, one has to identify a corresponding underlying which typically does not lead to a low-dimensional Markovian evolution.

Wouldn't it be nice to have a direct and simple connection between the observed implied volatilities and prices of more complex path-dependent contracts! Our objective is to provide such a linkage. This is done by using a combination of singular and regular perturbation techniques corresponding respectively to fast and slow time scales in volatility. We obtain a parametrization of the implied volatility surface in terms of Greeks, which involves four parameters at the first order of approximation. This procedure leads to parameters which are exactly those needed to price other contracts at this level of approximation. In our previous work presented in Fouque *et al.* (2000), we used only the fast volatility time scale combined with a statistical estimation of an effective constant volatility from historical data. The introduction of the slow volatility time scale enables us to capture more accurately the behavior of the term structure of implied volatility at long maturities. Yet, we preserve a parsimonious parametrization which effectively and robustly captures the main effects of time scale heterogeneity. Moreover, in the framework presented here, statistics of historical data are not needed for the calibration of these parameters.

Thus, in summary, we directly link the implied volatilities to prices of path-dependent contracts by exploiting volatility time scales. Furthermore, we extend this approach to interest rate and credit derivatives.

In Chapter 1 we review the basic ideas and methods of the Black–Scholes theory as well as the tools of stochastic calculus underpinning the models used. Chapter 2 provides a general introduction to stochastic volatility models. In Chapter 3, we identify time scales in financial data and introduce them in stochastic volatility models. In Chapter 4 we present the first-order perturbation theory in the context of European equity derivatives and identify the important parameters arising in this asymptotic analysis. This is the central chapter on the mathematical tools used in our multiscale modeling approach. In Chapter 5 we provide a calibration procedure for these parameters using observed implied volatilities. Indeed, these are the parameters that provide a parsimonious linkage between various contracts. We also show in this chapter how to extend the perturbation techniques to the case with time-dependent parameters needed for practical fitting of the presented S&P 500 data. The extensions to exotic and American claims are described in Chapters 6 and 7. It is also natural to exploit the presence of a skew of implied volatilities for designing hedging strategies of part of the volatility risk by trading the underlying. This is achieved in Chapter 8 by using the asymptotic analysis presented in the previous chapters combined with a martingale argument, which in turns can be used to derive asymptotics in the case of non-Markovian models of volatility. In Chapter 9 we present several extensions to the perturbation theory, including the cases with dividends and varying interest rates, and the derivation of the second-order corrections. Next, in Chapter 10, we discuss the Heston model, which is very popular for its computational tractability. We implement our perturbation theory on this particular model, we show how to generalize it while retaining its tractability, and we derive large deviation results in the regime of short maturities and fast mean-reverting volatility. Applications to variance-reduction techniques for Monte Carlo simulations, to portfolio optimization, and to estimation of CAPM Beta parameters are presented in Chapter 11. After introducing the basics of fixed income markets, we demonstrate in Chapter 12 that our perturbation approach is also effective for interest rates models with stochastic volatility. Then, we introduce the fundamental concepts used in credit risk modeling, and we apply our method to both single-name and multiname credit derivatives using structural models in Chapter 13 and intensity-based models in Chapter 14.

One cannot write a book in 2011 on financial mathematics without commenting on the recent financial crisis. We choose to do so in the Epilogue – Chapter 15 – since it involves judgement and behavior of the market players rather than mathematical modeling as presented in this book.