

Cambridge University Press

978-0-521-84263-1 - Modern Canonical Quantum General Relativity

Thomas Thiemann

Frontmatter

[More information](#)MODERN CANONICAL QUANTUM
GENERAL RELATIVITY

Modern physics rests on two fundamental building blocks: general relativity and quantum theory. General relativity is a geometric interpretation of gravity, while quantum theory governs the microscopic behaviour of matter. According to Einstein's equations, geometry is curved when and where matter is localized. Therefore, in general relativity, geometry is a dynamical quantity that cannot be prescribed *a priori* but is in interaction with matter. The equations of nature are background independent in this sense; there is no space-time geometry on which matter propagates without backreaction of matter on geometry. Since matter is described by quantum theory, which in turn couples to geometry, we need a quantum theory of gravity. The absence of a viable quantum gravity theory to date is due to the fact that quantum (field) theory as currently formulated assumes that a background geometry is available, thus being inconsistent with the principles of general relativity. In order to construct quantum gravity, one must reformulate quantum theory in a background-independent way. *Modern Canonical Quantum General Relativity* is about one such candidate for a background-independent quantum gravity theory: loop quantum gravity.

This book provides a complete treatise of the canonical quantization of general relativity. The focus is on detailing the conceptual and mathematical framework, describing the physical applications, and summarizing the status of this programme in its most popular incarnation: loop quantum gravity. Mathematical concepts and their relevance to physics are provided within this book, so it is suitable for graduate students and researchers with a basic knowledge of quantum field theory and general relativity.

THOMAS THIEMANN is Staff Scientist at the Max Planck Institut für Gravitationsphysik (Albert Einstein Institut), Potsdam, Germany. He is also a long-term researcher at the Perimeter Institute for Theoretical Physics and Associate Professor at the University of Waterloo, Canada. Thomas Thiemann obtained his Ph.D. in theoretical physics from the Rheinisch-Westfälisch Technische Hochschule, Aachen, Germany. He held two-year postdoctoral positions at The Pennsylvania State University and Harvard University. As of 2005 he holds a guest professor position at Beijing Normal University, China.

Cambridge University Press

978-0-521-84263-1 - Modern Canonical Quantum General Relativity

Thomas Thiemann

Frontmatter

[More information](#)

CAMBRIDGE MONOGRAPHS ON MATHEMATICAL PHYSICS

General editors: P. V. Landshoff, D. R. Nelson, S. Weinberg

- S. J. Aarseth *Gravitational N-Body Simulations*
- J. Ambjørn, B. Durhuus and T. Jonsson *Quantum Geometry: A Statistical Field Theory Approach*
- A. M. Anile *Relativistic Fluids and Magneto-Fluids: With Applications in Astrophysics and Plasma Physics*
- J. A. de Azcárrage and J. M. Izquierdo *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics*[†]
- O. Babelon, D. Bernard and M. Talon *Introduction to Classical Integrable Systems*[†]
- F. Bastianelli and P. van Nieuwenhuizen *Path Integrals and Anomalies in Curved Space*
- V. Belinskii and E. Verdaguer *Gravitational Solitons*
- J. Bernstein *Kinetic Theory in the Expanding Universe*
- G. F. Bertsch and R. A. Broglia *Oscillations in Finite Quantum Systems*
- N. D. Birrell and P. C. W. Davies *Quantum Fields in Curved space*[†]
- M. Burgess *Classical Covariant Fields*
- S. Carlip *Quantum Gravity in 2 + 1 Dimensions*[†]
- P. Cartier and C. DeWitt-Morette *Functional Integration: Action and Symmetries*
- J. C. Collins *Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion*[†]
- M. Creutz *Quarks, Gluons and Lattices*[†]
- P. D. D'Eath *Supersymmetric Quantum Cosmology*
- F. de Felice and C. J. S. Clarke *Relativity on Curved Manifolds*[†]
- B. S. DeWitt *Supermanifolds, 2nd edition*[†]
- P. G. O. Freund *Introduction to Supersymmetry*[†]
- J. Fuchs *Affine Lie Algebras and Quantum Groups: An Introduction, with Applications in Conformal Field Theory*[†]
- J. Fuchs and C. Schweigert *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists*[†]
- Y. Fujii and K. Maeda *The Scalar-Tensor Theory of Gravitation*
- A. S. Galperin, E. A. Ivanov, V. I. Orievetsky and E. S. Sokatchev *Harmonic Superspace*
- R. Gambini and J. Pullin *Loops, Knots, Gauge Theories and Quantum Gravity*[†]
- T. Gannon *Moonshine Beyond the Monster: The Bridge Connecting Algebra, Modular Forms and Physics*
- M. Göckeler and T. Schücker *Differential Geometry, Gauge Theories and Gravity*[†]
- C. Gómez, M. Ruiz-Altaba and G. Sierra *Quantum Groups in Two-dimensional Physics*
- M. B. Green, J. H. Schwarz and E. Witten *Superstring Theory, Volume 1: Introduction*[†]
- M. B. Green, J. H. Schwarz and E. Witten *Superstring Theory, Volume 2: Loop Amplitudes, Anomalies and Phenomenology*[†]
- V. N. Gribov *The Theory of Complex Angular Momenta: Gribov Lectures on Theoretical Physics*
- S. W. Hawking and G. F. R. Ellis *The Large-Scale Structure of Space-Time*[†]
- F. Iachello and A. Arima *The Interacting Boson Model*
- F. Iachello and P. van Isacker *The Interacting Boson-Fermion Model*
- C. Itzykson and J.-M. Drouffe *Statistical Field Theory, Volume 1: From Brownian Motion to Renormalization and Lattice Gauge Theory*[†]
- C. Itzykson and J.-M. Drouffe *Statistical Field Theory, Volume 2: Strong Coupling, Monte Carlo Methods, Conformal Field Theory, and Random Systems*[†]
- C. Johnson *D-Branes*[†]
- J. I. Kapusta and C. Gale *Finite-Temperature Field Theory, 2nd edition*
- V. E. Korepin, A. G. Izergin and N. M. Bogoliubov *The Quantum Inverse Scattering Method and Correlation Functions*
- M. Le Bellac *Thermal Field Theory*[†]
- Y. Makeenko *Methods of Contemporary Gauge Theory*
- N. Manton and P. Sutcliffe *Topological Solitons*
- N. H. March *Liquid Metals: Concepts and Theory*
- I. M. Montvay and G. Münster *Quantum Fields on a Lattice*[†]
- L. O'Riadaigh *Group Structure of Gauge Theories*[†]
- T. Ort 'in *Gravity and Strings*
- A. Ozorio de Almeida *Hamiltonian Systems: Chaos and Quantization*[†]
- R. Penrose and W. Rindler *Spinors and Space-Time, Volume 1: Two-Spinor Calculus and Relativistic Fields*[†]
- R. Penrose and W. Rindler *Spinors and Space-Time, Volume 2: Spinor and Twistor Methods in Space-Time Geometry*[†]
- S. Pokorski *Gauge Field Theories, 2nd edition*
- J. Polchinski *String Theory, Volume 1: An Introduction to the Bosonic String*
- J. Polchinski *String Theory, Volume 2: Superstring Theory and Beyond*
- V. N. Popov *Functional Integrals and Collective Excitations*[†]
- R. J. Rivers *Path Integral Methods in Quantum Field Theory*[†]
- R. G. Roberts *The Structure of the Proton: Deep Inelastic Scattering*[†]
- C. Rovelli *Quantum Gravity*

Cambridge University Press

978-0-521-84263-1 - Modern Canonical Quantum General Relativity

Thomas Thiemann

Frontmatter

[More information](#)

W. C. Saslaw *Gravitational Physics of Stellar and Galactic Systems*[†]

H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers and E. Herlt *Exact Solutions of Einstein's Field Equations, 2nd edition*

J. M. Stewart *Advanced General Relativity*[†]

T. Thiemann *Modern Canonical Quantum General Relativity*

A. Vilenkin and E. P. S. Shellard *Cosmic Strings and Other Topological Defects*[†]

R. S. Ward and R. O. Wells Jr *Twistor Geometry and Field Theory*[†]

J. R. Wilson and G. J. Mathews *Relativistic Numerical Hydrodynamics*

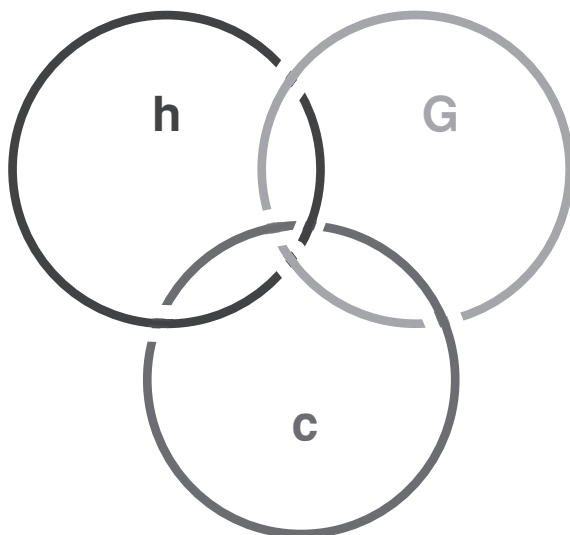
[†]Issued as a paperback

Cambridge University Press
978-0-521-84263-1 - Modern Canonical Quantum General Relativity
Thomas Thiemann
Frontmatter
[More information](#)

Modern Canonical Quantum General Relativity

THOMAS THIEMANN

Max Planck Institut für Gravitationsphysik, Germany



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press
978-0-521-84263-1 - Modern Canonical Quantum General Relativity
Thomas Thiemann
Frontmatter
[More information](#)

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo
Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK
Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521842631

© T. Thiemann 2007

This publication is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 2007

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-84263-1 hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for
external or third-party internet websites referred to in this publication, and does not
guarantee that any content on such websites is, or will remain, accurate or appropriate.

Cambridge University Press

978-0-521-84263-1 - Modern Canonical Quantum General Relativity

Thomas Thiemann

Frontmatter

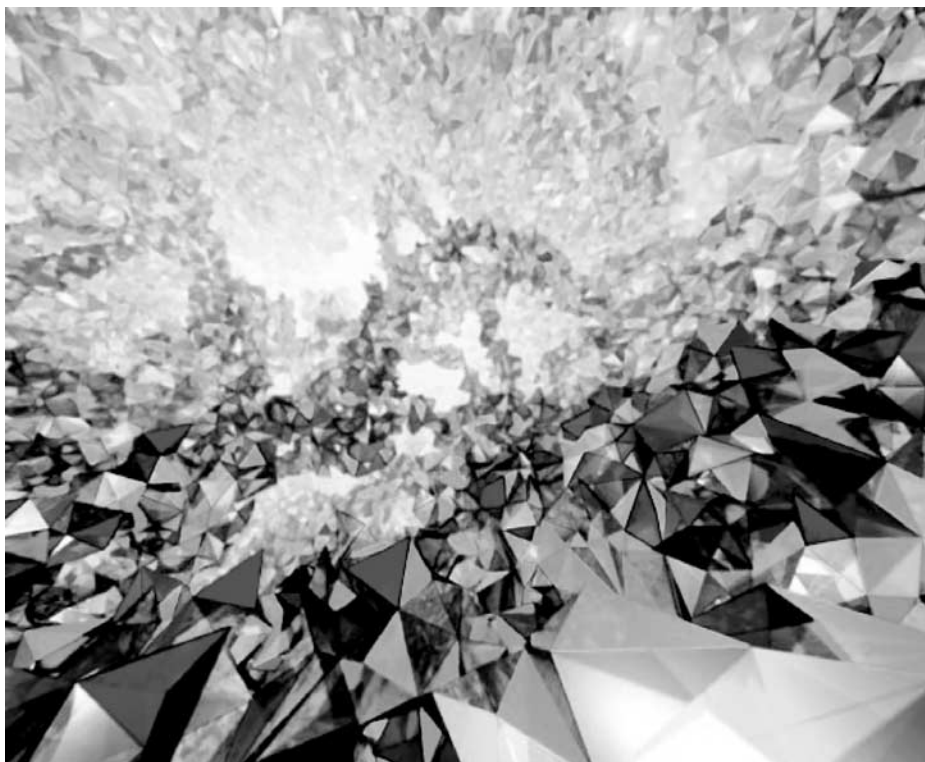
[More information](#)

Figure 1 Copyright: Max Planck Institute for Gravitational Physics (Albert Einstein Institute), MildeMarketing Science Communication, Exozet. To see the animation, please visit the URL <http://www.einstein-online.info/de/vertiefung/Spinnetzwerke/index.html>.

Quantum spin dynamics

This is a still from an animation which illustrates the dynamical evolution of quantum geometry in Loop Quantum Gravity (LQG), which is a particular incarnation of canonical Quantum General Relativity.

The faces of the tetrahedra are elementary excitations (atoms) of geometry. Each face is coloured, where red and violet respectively means that the face carries low or high area respectively. The colours or areas are quantised in units of the Planck area $\ell_P \approx 10^{-7} \text{ cm}^2$. Thus the faces do not have area as they appear to have in the figure, rather one would have to shrink red and stretch violet faces accordingly in order to obtain the correct picture.

The faces are dual to a four-valent graph, that is, each face is punctured by an edge which connects the centres of the tetrahedra with a common face. These edges are ‘charged’ with half-integral spin-quantum numbers and these numbers are proportional to the quantum area of the faces. The collection of spins and edges defines a spin-network state. The spin quantum numbers are created and annihilated at each Planck time step of $\tau_P \approx 10^{-43} \text{ s}$ in a specific way as dictated by the quantum Einstein equations. Hence the name Quantum Spin Dynamics (QSD) in analogy to Quantum Chromodynamics (QCD).

Spin zero corresponds to no edge or face at all, hence whole tetrahedra are created and annihilated all the time. Therefore, the free space not occupied by tetrahedra does not correspond to empty (matter-free) space but rather to space without geometry, it has zero volume and therefore is a hole in the quantum spacetime. *The tetrahedra are not embedded in space, they are the space.* Matter can only exist where geometry is excited, that is, on the edges (bosons) and vertices (fermions) of the graph. Thus geometry is completely discrete and chaotic at the Planck scale, only on large scales does it appear smooth.

In this book, this fascinating physics is explained in mathematical detail.

Contents

<i>Foreword, by Chris Isham</i>	<i>page</i> xvii
<i>Preface</i>	xix
<i>Notation and conventions</i>	xxiii
Introduction: Defining quantum gravity	1
Why quantum gravity in the twenty-first century?	1
The role of background independence	8
Approaches to quantum gravity	11
Motivation for canonical quantum general relativity	23
Outline of the book	25
I CLASSICAL FOUNDATIONS, INTERPRETATION AND THE CANONICAL QUANTISATION PROGRAMME	
1 Classical Hamiltonian formulation of General Relativity	39
1.1 The ADM action	39
1.2 Legendre transform and Dirac analysis of constraints	46
1.3 Geometrical interpretation of the gauge transformations	50
1.4 Relation between the four-dimensional diffeomorphism group and the transformations generated by the constraints	56
1.5 Boundary conditions, gauge transformations and symmetries	60
1.5.1 Boundary conditions	60
1.5.2 Symmetries and gauge transformations	65
2 The problem of time, locality and the interpretation of quantum mechanics	74
2.1 The classical problem of time: Dirac observables	75
2.2 Partial and complete observables for general constrained systems	81
2.2.1 Partial and weak complete observables	82
2.2.2 Poisson algebra of Dirac observables	85
2.2.3 Evolving constants	89
2.2.4 Reduced phase space quantisation of the algebra of Dirac observables and unitary implementation of the multi-fingered time evolution	90
2.3 Recovery of locality in General Relativity	93

x	<i>Contents</i>	
2.4	Quantum problem of time: physical inner product and interpretation of quantum mechanics	95
2.4.1	Physical inner product	95
2.4.2	Interpretation of quantum mechanics	98
3	The programme of canonical quantisation	107
3.1	The programme	108
4	The new canonical variables of Ashtekar for General Relativity	118
4.1	Historical overview	118
4.2	Derivation of Ashtekar's variables	123
4.2.1	Extension of the ADM phase space	123
4.2.2	Canonical transformation on the extended phase space	126
 II FOUNDATIONS OF MODERN CANONICAL QUANTUM GENERAL RELATIVITY		
5	Introduction	141
5.1	Outline and historical overview	141
6	Step I: the holonomy–flux algebra \mathfrak{P}	157
6.1	Motivation for the choice of \mathfrak{P}	157
6.2	Definition of \mathfrak{P} : (1) Paths, connections, holonomies and cylindrical functions	162
6.2.1	Semianalytic paths and holonomies	162
6.2.2	A natural topology on the space of generalised connections	168
6.2.3	Gauge invariance: distributional gauge transformations	175
6.2.4	The C^* algebraic viewpoint and cylindrical functions	183
6.3	Definition of \mathfrak{P} : (2) surfaces, electric fields, fluxes and vector fields	191
6.4	Definition of \mathfrak{P} : (3) regularisation of the holonomy–flux Poisson algebra	194
6.5	Definition of \mathfrak{P} : (4) Lie algebra of cylindrical functions and flux vector fields	202
7	Step II: quantum $*$-algebra \mathfrak{A}	206
7.1	Definition of \mathfrak{A}	206
7.2	(Generalised) bundle automorphisms of \mathfrak{A}	209
8	Step III: representation theory of \mathfrak{A}	212
8.1	General considerations	212
8.2	Uniqueness proof: (1) existence	219
8.2.1	Regular Borel measures on the projective limit: the uniform measure	220
8.2.2	Functional calculus on a projective limit	226

8.2.3	+ Density and support properties of $\mathcal{A}, \mathcal{A}/\mathcal{G}$ with respect to $\overline{\mathcal{A}}, \overline{\mathcal{A}/\mathcal{G}}$	233
8.2.4	Spin-network functions and loop representation	237
8.2.5	Gauge and diffeomorphism invariance of μ_0	242
8.2.6	+ Ergodicity of μ_0 with respect to spatial diffeomorphisms	245
8.2.7	Essential self-adjointness of electric flux momentum operators	246
8.3	Uniqueness proof: (2) uniqueness	247
8.4	Uniqueness proof: (3) irreducibility	252
9	Step IV: (1) implementation and solution of the kinematical constraints	264
9.1	Implementation of the Gauß constraint	264
9.1.1	Derivation of the Gauß constraint operator	264
9.1.2	Complete solution of the Gauß constraint	266
9.2	Implementation of the spatial diffeomorphism constraint	269
9.2.1	Derivation of the spatial diffeomorphism constraint operator	269
9.2.2	General solution of the spatial diffeomorphism constraint	271
10	Step IV: (2) implementation and solution of the Hamiltonian constraint	279
10.1	Outline of the construction	279
10.2	Heuristic explanation for UV finiteness due to background independence	282
10.3	Derivation of the Hamiltonian constraint operator	286
10.4	Mathematical definition of the Hamiltonian constraint operator	291
10.4.1	Concrete implementation	291
10.4.2	Operator limits	296
10.4.3	Commutator algebra	300
10.4.4	The quantum Dirac algebra	309
10.5	The kernel of the Wheeler–DeWitt constraint operator	311
10.6	The Master Constraint Programme	317
10.6.1	Motivation for the Master Constraint Programme in General Relativity	317
10.6.2	Definition of the Master Constraint	320
10.6.3	Physical inner product and Dirac observables	326
10.6.4	Extended Master Constraint	329
10.6.5	Algebraic Quantum Gravity (AQG)	331
10.7	+ Further related results	334
10.7.1	The Wick transform	334
10.7.2	Testing the new regularisation technique by models of quantum gravity	340

xii	<i>Contents</i>	
	10.7.3 Quantum Poincaré algebra	341
	10.7.4 Vasiliev invariants and discrete quantum gravity	344
11	Step V: semiclassical analysis	345
11.1	+ Weaves	349
11.2	Coherent states	353
	11.2.1 Semiclassical states and coherent states	354
	11.2.2 Construction principle: the complexifier method	356
	11.2.3 Complexifier coherent states for diffeomorphism-invariant theories of connections	362
	11.2.4 Concrete example of complexifier	367
	11.2.5 Semiclassical limit of loop quantum gravity: graph-changing operators, shadows and diffeomorphism-invariant coherent states	376
	11.2.6 + The infinite tensor product extension	385
11.3	Graviton and photon Fock states from $L_2(\overline{\mathcal{A}}, d\mu_0)$	390
III PHYSICAL APPLICATIONS		
12	Extension to standard matter	399
12.1	The classical standard model coupled to gravity	400
	12.1.1 Fermionic and Einstein contribution	401
	12.1.2 Yang–Mills and Higgs contribution	405
12.2	Kinematical Hilbert spaces for diffeomorphism-invariant theories of fermion and Higgs fields	406
	12.2.1 Fermionic sector	406
	12.2.2 Higgs sector	411
	12.2.3 Gauge and diffeomorphism-invariant subspace	417
12.3	Quantisation of matter Hamiltonian constraints	418
	12.3.1 Quantisation of Einstein–Yang–Mills theory	419
	12.3.2 Fermionic sector	422
	12.3.3 Higgs sector	425
	12.3.4 A general quantisation scheme	429
13	Kinematical geometrical operators	431
13.1	Derivation of the area operator	432
13.2	Properties of the area operator	434
13.3	Derivation of the volume operator	438
13.4	Properties of the volume operator	447
	13.4.1 Cylindrical consistency	447
	13.4.2 Symmetry, positivity and self-adjointness	448
	13.4.3 Discreteness and anomaly-freeness	448
	13.4.4 Matrix elements	449
13.5	Uniqueness of the volume operator, consistency with the flux operator and pseudo-two-forms	453

Contents xiii

13.6	Spatially diffeomorphism-invariant volume operator	455
14	Spin foam models	458
14.1	Heuristic motivation from the canonical framework	458
14.2	Spin foam models from BF theory	462
14.3	The Barrett–Crane model	466
14.3.1	Plebanski action and simplicity constraints	466
14.3.2	Discretisation theory	472
14.3.3	Discretisation and quantisation of BF theory	476
14.3.4	Imposing the simplicity constraints	482
14.3.5	Summary of the status of the Barrett–Crane model	494
14.4	Triangulation dependence and group field theory	495
14.5	Discussion	502
15	Quantum black hole physics	511
15.1	Classical preparations	514
15.1.1	Null geodesic congruences	514
15.1.2	Event horizons, trapped surfaces and apparent horizons	517
15.1.3	Trapping, dynamical, non-expanding and (weakly) isolated horizons	519
15.1.4	Spherically symmetric isolated horizons	526
15.1.5	Boundary symplectic structure for SSIHs	535
15.2	Quantisation of the surface degrees of freedom	540
15.2.1	Quantum U(1) Chern–Simons theory with punctures	541
15.3	Implementing the quantum boundary condition	546
15.4	Implementation of the quantum constraints	548
15.4.1	Remaining U(1) gauge transformations	549
15.4.2	Remaining surface diffeomorphism transformations	550
15.4.3	Final physical Hilbert space	550
15.5	Entropy counting	550
15.6	Discussion	557
16	Applications to particle physics and quantum cosmology	562
16.1	Quantum gauge fixing	562
16.2	Loop Quantum Cosmology	563
17	Loop Quantum Gravity phenomenology	572

IV MATHEMATICAL TOOLS AND THEIR CONNECTION TO PHYSICS

18	Tools from general topology	577
18.1	Generalities	577
18.2	Specific results	581

19 Differential, Riemannian, symplectic and complex geometry	585
19.1 Differential geometry	585
19.1.1 Manifolds	585
19.1.2 Passive and active diffeomorphisms	587
19.1.3 Differential calculus	590
19.2 Riemannian geometry	606
19.3 Symplectic manifolds	614
19.3.1 Symplectic geometry	614
19.3.2 Symplectic reduction	616
19.3.3 Symplectic group actions	621
19.4 Complex, Hermitian and Kähler manifolds	623
20 Semianalytic category	627
20.1 Semianalytic structures on \mathbb{R}^n	627
20.2 Semianalytic manifolds and submanifolds	631
21 Elements of fibre bundle theory	634
21.1 General fibre bundles and principal fibre bundles	634
21.2 Connections on principal fibre bundles	636
22 Holonomies on non-trivial fibre bundles	644
22.1 The groupoid of equivariant maps	644
22.2 Holonomies and transition functions	647
23 Geometric quantisation	652
23.1 Prequantisation	652
23.2 Polarisation	662
23.3 Quantisation	668
24 The Dirac algorithm for field theories with constraints	671
24.1 The Dirac algorithm	671
24.2 First- and second-class constraints and the Dirac bracket	674
25 Tools from measure theory	680
25.1 Generalities and the Riesz–Markov theorem	680
25.2 Measure theory and ergodicity	687
26 Key results from functional analysis	689
26.1 Metric spaces and normed spaces	689
26.2 Hilbert spaces	691
26.3 Banach spaces	693
26.4 Topological spaces	694
26.5 Locally convex spaces	694
26.6 Bounded operators	695
26.7 Unbounded operators	697

<i>Contents</i>	xv
26.8 Quadratic forms	699
27 Elementary introduction to Gel'fand theory for Abelian C^*-algebras	701
27.1 Banach algebras and their spectra	701
27.2 The Gel'fand transform and the Gel'fand isomorphism	709
28 Bohr compactification of the real line	713
28.1 Definition and properties	713
28.2 Analogy with loop quantum gravity	715
29 Operator $*$-algebras and spectral theorem	719
29.1 Operator $*$ -algebras, representations and GNS construction	719
29.2 Spectral theorem, spectral measures, projection valued measures, functional calculus	723
30 Refined algebraic quantisation (RAQ) and direct integral decomposition (DID)	729
30.1 RAQ	729
30.2 Master Constraint Programme (MCP) and DID	735
31 Basics of harmonic analysis on compact Lie groups	746
31.1 Representations and Haar measures	746
31.2 The Peter and Weyl theorem	752
32 Spin-network functions for $SU(2)$	755
32.1 Basics of the representation theory of $SU(2)$	755
32.2 Spin-network functions and recoupling theory	757
32.3 Action of holonomy operators on spin-network functions	762
32.4 Examples of coherent state calculations	765
33 + Functional analytic description of classical connection dynamics	770
33.1 Infinite-dimensional (symplectic) manifolds	770
<i>References</i>	775
<i>Index</i>	809

Foreword

Over half a century of collective study has not diminished the fascination of searching for a consistent theory of quantum gravity. I first encountered the subject in 1969 when, as a young researcher, I spent a year in Trieste working with Abdus Salam who, for a while, was very interested in the subject. In those days, the technical approaches adopted for quantum gravity depended very much on the background of the researcher: those, like myself, from a theoretical particle-physics background used perturbative quantum field theory; those whose background was in general relativity tended to use relatively elementary quantum theory, but taking full account of the background general relativity (which the other scheme did not).

The perturbative quantum field theory schemes foundered on intractable ultraviolet divergences and gave way to super-gravity – the super-symmetric extension of standard general relativity. In spite of initial optimism, this approach succumbed to the same disease and was eventually replaced by the far more ambitious superstring theories. Superstring theory is now the dominant quantum gravity programme in terms of the number of personnel involved and the number of published papers, per year, per unit researcher.

However, notwithstanding my early training as a quantum field theorist, I quickly became fascinated by the “canonical quantization”, or “quantum geometry,” schemes favored by those coming from general relativity. The early attempts for quantizing the metric variables were rather naive, and took on various forms according to how the intrinsic constraints of classical general relativity are handled. In the most popular approach, the constraints are imposed on the state vectors and give rise to the famous Wheeler–DeWitt equation – arguably one of the most elegant equations in theoretical physics, and certainly one of the most mathematically ill-defined. Indeed, it was the very intractability of this equation that first intrigued me and prompted me to see what could be done with more sophisticated quantization methods. After much effort it became clear that the answer was “not much.”

The enormous difficulty of the canonical quantum gravity scheme eventually caused it to go into something of a decline, until new life was imparted with Ashtekar’s discovery of a set of variables in which the constraint equations simplify significantly. This scheme slowly morphed into “loop quantum gravity:” an approach which has, for the first time, allowed real insight into what a non-perturbative quantisation of general relativity might look like. A number of

genuine results were obtained, but it became slowly apparent that the old problems with the Wheeler–DeWitt equation were still there in transmuted form, and the critical Hamiltonian constraint was still ill-defined.

It was at this point that Thomas Thiemann – the author of this book – entered the scene. I can still remember the shock I felt when I first read the papers he put onto the web dealing with the Hamiltonian constraint. Suddenly, someone with a top-rate mathematical knowledge had addressed this critical question anew, and with considerable success. Indeed, Thiemann succeeded with loop quantum gravity where I had failed with the old Wheeler–DeWitt equation, and he has gone on since that time to become one of the internationally acknowledged experts in loop quantum gravity.

Thiemann’s deep knowledge of mathematics applied to quantum gravity is evident from the first page of this magnificent book. The subject is explored in considerable generality and with real mathematical depth. The author starts from first principles with a general introduction to quantum gravity, and then proceeds to give, what is by far, the most comprehensive, and mathematically precise, exposition of loop quantum gravity that is available in the literature. The reader should be warned though that, when it comes to mathematics, the author takes no hostages, and a good knowledge of functional analysis and differential geometry is assumed from the outset. Still, that is how the subject is these days, and anyone who seriously aspires to work in loop quantum gravity would be advised to gain a good knowledge of this type of mathematics. In that sense, this is a text that is written for advanced graduate students, or professionals who work in the area.

My graduate students not infrequently ask me what I think of the current status of canonical quantum gravity and, in particular, what I think the chances are of ever making proper mathematical sense of the constraints that define the theory. For some years now I have replied to the effect that, if anybody can do it, it will be Thomas Thiemann and, if he cannot do it, then probably nobody will. Anyone who reads right through this major new work will understand why I place so much trust in the author’s ability to crack this central problem of quantum gravity.

Chris Isham,
Professor of Theoretical Physics at
The Blankett Laboratory, Imperial College, London

Preface

Quantum General Relativity (QGR) or Quantum Gravity for short is, by definition, a Quantum (Field) Theory of Einstein's geometrical interpretation of gravity which he himself called General Relativity (GR). It is a theory which synthesises the two fundamental building blocks of modern physics, that is, (1) the generally relativistic principle of background independence, sometimes called general covariance and (2) the uncertainty principle of quantum mechanics.

The search for a viable QGR theory is almost as old as Quantum Mechanics and GR themselves, however, despite an enormous effort of work by a vast amount of physicists over the past 70 years, we still do not have a credible QGR theory. Since the problem is so hard, QGR is sometimes called the 'holy grail of physics'. Indeed, it is to be expected that the discovery of a QGR theory revolutionises our current understanding of nature in a way as radical as both General Relativity and Quantum Mechanics did.

What we do have today are candidate theories which display some promising features that one intuitively expects from a quantum theory of gravity. They are so far candidates only because for each of them one still has to show, at the end of the construction of the theory, that it reduces to the presently known standard model of matter and classical General Relativity at low energies, which is the minimal test that any QGR theory must pass.

One of these candidates is Loop Quantum Gravity (LQG). LQG is a modern version of the canonical or Hamiltonian approach to Quantum Gravity, originally introduced by Dirac, Bergmann, Komar, Wheeler, DeWitt, Arnowitt, Deser and Misner. It is modern in the sense that the theory is formulated in terms of connections ('gauge potentials') rather than metrics. It is due to this fact that the theory was called Loop Quantum Gravity since theories of connections are naturally described in terms of Wilson loops. This also brings GR much closer to the formulation of the other three forces of nature, each of which is described in terms of connections of a particular Yang–Mills theory for which viable quantum theories exist. Consequently, the connection reformulation has resulted in rapid progress over the past 20 years.

The purpose of this book is to provide a self-contained treatise on canonical – and in particular Loop Quantum Gravity. Although the theory is still under rapid development and the present book therefore is at best a snapshot, the field has now matured enough in order to justify the publication of a new textbook. The literature on LQG now comprises more than a thousand

Cambridge University Press

978-0-521-84263-1 - Modern Canonical Quantum General Relativity

Thomas Thiemann

Frontmatter

[More information](#)

xx

Preface

articles scattered over a vast number of journals, reviews, proceedings and conference reports. Structures which were believed to be essential initially turned out to be negligible later on and vice versa, thus making it very hard for the beginner to get an overview of the subject. We hope that this book serves as a ‘geodesic’ through the literature enabling the reader to move quickly from the basics to the frontiers of current research. By definition, a geodesic cannot touch on all the subjects of the theory and we apologise herewith to our colleagues if we were unable to cover their work in this single volume manuscript. However, guides to further reading and a detailed bibliography try to compensate for this incompleteness. A complete listing of all LQG-related papers, which is periodically being updated, can be found in [1, 2].¹

Loop Quantum Gravity is an attempt to construct a mathematically rigorous, background-independent, non-perturbative Quantum Field Theory of Lorentzian General Relativity and all known matter in four spacetime dimensions, not more and not less. In particular, no claim is made that LQG is a unified theory of everything predicting, among other things, matter content and dimensionality of the world. Hence, currently there is no restriction on the allowed matter couplings although these might still come in at a later stage when deriving the low energy limit. While the connection formulation works only in four spacetime dimensions and in that sense is a prediction, higher p -form formulations in higher dimensions are conceivable. Matter and geometry are not unified in the sense that they are components of one and the same geometrical object, however, they are unified under the four-dimensional diffeomorphism group which in perturbative approaches is broken. LQG provides a universal framework for how to combine quantum theory and General Relativity for all possible matter and in that sense is robust against the very likely discovery of further substructure of matter between the energy scales of the LHC and the Planck scale which differ by 16 orders of magnitude. This is almost the same number of orders of magnitude as between 1 mm and the length scales that the LHC can resolve, and we found a huge amount of substructure there.

The stress on mathematical rigour is here no luxurious extra baggage but a necessity: in a field where, to date, no experimental input is available, mathematical consistency is the only guiding principle to construct the theory. The strategy is to combine the presently known physical principles and to drive them to their logical frontiers without assuming any extra, unobserved structure such as extra dimensions and extra particles. This deliberately conservative approach has the advantage of either producing a viable theory or of deriving which extra structures are needed in order to produce a successful theory. Indeed, it is conceivable that at some point in the development of the theory a ‘quantum leap’ is necessary, similar to Heisenberg’s discovery that the

See also the URLs <http://www.nucleares.unam.mx/corichi/lqgbib.pdf> and <http://www.matmor.unam.mx/corichi/lqgbib.pdf>.

Bohr–Sommerfeld quantisation rules can be interpreted in terms of operators. The requirement to preserve background independence has already led to new, fascinating mathematical structures. For instance, a fundamental discreteness of spacetime at the Planck scale of 10^{-33} cm seems to be a prediction of the theory which is a first substantial evidence for a theory in which the gravitational field acts as a natural cutoff of the usual ultraviolet divergences of QFT.

Accordingly, the present text tries to be mathematically precise. We will develop in depth the conceptual and mathematical framework underlying LQG, stating exact definitions and theorems including complete proofs. Many of the calculations or arguments used during the proofs cannot be found anywhere in the literature detailed as they are displayed here. We have supplied a vast amount of mathematical background information so that the book can be read by readers with only basic prior knowledge of GR and QFT without having to consult too much additional literature. We have made an effort to stress the basic principles of canonical QGR, of which LQG is just one possible incarnation based on a specific choice of variables.

For readers who want to get acquainted first with the physical ideas and conceptual aspects of LQG before going into mathematical details, we strongly recommend the book by Carlo Rovelli [3]. The two books are complementary in the sense that they can be regarded almost as Volume I (‘Introduction and Conceptual Framework’) and Volume II (‘Mathematical Framework and Applications’) of a general presentation of QGR in general and LQG in particular. While this book also develops a tight conceptual framework, the book by Carlo Rovelli is much broader in that aspect. Recent review articles can be found in [4–14]. The status of the theory a decade ago is summarised in the books [15–17].

The present text is aimed at all readers who want to find out in detail how LQG works, conceptually and technically, enabling them to quickly develop their own research on the subject. For instance, the author taught most of the material of this book in a two-semester course to German students in physics and mathematics who were in their sixth semester of diploma studies or higher. After that they could complete diploma theses or PhD theses on the subject without much further guidance. Unfortunately, due to reasons of space, exercises and their solutions had to be abandoned from the book, see [12] for a selection. We hope to incorporate them in an extended future edition. As we have pointed out, LQG is far from being a completed theory and aspects of LQG which are at the frontier of current research and whose details are still under construction will be critically discussed. This will help readers to get an impression of what important open problems there are and hopefully encourage them to address these in their own research.

The numerous suggestions for improvements to the previous online version of this book (<http://www.arxiv.org/list/gr-qc/0110034>) by countless colleagues is gratefully acknowledged, in particular those by Jürgen Ehlers, Christian Fleischhack, Stefan Hofmann, Chris Isham, Jurek Lewandowski, Robert Oeckl, Hendryk Pfeiffer, Carlo Rovelli, Hanno Sahlmann and Oliver Winkler. Special

Cambridge University Press
978-0-521-84263-1 - Modern Canonical Quantum General Relativity
Thomas Thiemann
Frontmatter
[More information](#)

xxii

Preface

thanks go to my students Johannes Brunnemann, Bianca Dittrich and Kristina Giesel for a careful reading of the manuscript and especially to Kristina Giesel for her help with the figures.

Посвящаю своей жене Татьяне.

Ebenso gewidmet meinen Söhnen Andreas und Maximilian.

Thomas Thiemann
Berlin, Toronto 2001–2007

Notation and conventions

Symbol	Meaning
$G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$	Newton's constant
$\kappa = 16\pi G/c^3$	gravitational coupling constant
$\ell_p = \sqrt{\hbar \kappa} \approx 10^{-33} \text{ cm}$	Planck length
$m_p = \sqrt{\hbar/\kappa}/c \approx 10^{19} \text{ GeV}/c^2$	Planck mass
Q	Yang–Mills coupling constant
$M, \dim(M) = D + 1$	spacetime manifold
$\sigma, \dim(\sigma) = D$	abstract spatial manifold
Σ	spatial manifold embedded into M
G	compact gauge group
$\text{Lie}(G)$	Lie algebra
$N - 1$	rank of gauge group
$\mu, \nu, \rho, .. = 0, 1, \dots, D$	tensorial spacetime indices
$a, b, c, .. = 1, \dots, D$	tensorial spatial indices
$\epsilon_{a_1 .. a_D}$	Levi–Civita totally skew tensor pseudo density of weight -1
$g_{\mu\nu}$	spacetime metric tensor
q_{ab}	spatial (intrinsic) metric tensor of σ
K_{ab}	extrinsic curvature of σ
R	curvature tensor
\underline{h}	group elements for general G
$\underline{h}_{mn}, m, n, o, .. = 1, \dots, N$	matrix elements for general G
$I, J, K, .. = 1, 2, \dots, \dim(G)$	Lie algebra indices for general G
$\underline{\tau}_I$	Lie algebra generators for general G
k_{IJ}	$= -\text{tr}(\underline{\tau}_I \underline{\tau}_J)/N := \delta_{IJ}$: Cartan–Killing metric for G
$[\underline{\tau}_I, \underline{\tau}_J] = 2f_{IJ}{}^K \underline{\tau}_K$	structure constants for G
$\underline{\pi}(\underline{h})$	(irreducible) representations for general G or algebra
h	group elements for $\text{SU}(2)$
$h_{AB}, A, B, C, .. = 1, 2$	matrix elements for $\text{SU}(2)$
$i, j, k, .. = 1, 2, 3$	Lie algebra indices for $\text{SU}(2)$
τ_i	Lie algebra generators for $\text{SU}(2)$

$k_{ij} = \delta_{ij}$	Cartan–Killing metric for SU(2)
$f_{ij}{}^k = \epsilon_{ijk}$	structure constants for SU(2)
$\pi_j(h)$	(irreducible) representations for SU(2) with spin j
\underline{A}	connection on G-bundle over σ
\underline{A}_a^I	pull-back of \underline{A} to σ by local section
$\iota_A, o_A; A = 1, 2$	$\phi_A \iota^A := \epsilon^{AB} \iota_B = 1$: spinor dyad
$\bar{\iota}_{A'}, \bar{o}_{A'}; A' = 1, 2$	primed (complex conjugate) spinor dyad
g	gauge transformation or element of complexification of G
P	principal G-bundle
A	connection on SU(2)-bundle over σ
A_a^i	pull-back of A to σ by local section
$*\underline{E}$	pseudo- $(D-1)$ -form in vector bundle associated to G-bundle under adjoint representation
$*\underline{E}_{a_1 \dots, a_{D-1}}^I$	$:= k^{IJ} \epsilon_{a_1 \dots, a_{D-1}} \underline{E}_J^{a_D}$: pull-back of $*\underline{E}$ to σ by local section
$*E$	pseudo- $(D-1)$ -form in vector bundle associated to SU(2)-bundle under adjoint representation
$*E_{a_1 \dots, a_{D-1}}^i$	$:= k^{ij} \epsilon_{a_1 \dots, a_{D-1}} E_j^{a_D}$: pull-back of $*E$ to σ by local section
E_j^a	$:= \epsilon^{a_1 \dots a_{D-1}} (*E)_{a_1 \dots a_{D-1}}^k k_{jk} / ((D-1)!)$: ‘electric fields’
e	one-form co-vector bundle associated to the SU(2)-bundle under the defining representation (D -bein)
e_a^i	pull-back of e to σ by local section
Γ_a^i	pull-back by local section of SU(2) spin connection over σ
R, X	right-invariant vector field on G
L	Left-invariant vector field on G
$Y = iX$	momentum vector field
\mathcal{M}	phase space
\mathcal{E}	Banach manifold or space of smooth electric fields
$T_{(a_1 \dots a_n)}$	$:= \frac{1}{n!} \sum_{\iota \in S_n} T_{a_{\iota(1)} \dots a_{\iota(n)}}$: symmetrisation of indices
$T_{[a_1 \dots a_n]}$	$:= \frac{1}{n!} \sum_{\iota \in S_n} \text{sgn}(\iota) T_{a_{\iota(1)} \dots a_{\iota(n)}}$: antisymmetrisation of indices
\mathcal{A}	space of smooth connections
\mathcal{G}	space of smooth gauge transformations

$\overline{\mathcal{A}}$	space of distributional connections
$\mathcal{A}^{\mathbb{C}}$	space of smooth complex connections
$\mathcal{G}^{\mathbb{C}}$	space of smooth complex gauge transformations
$\overline{\mathcal{G}}$	space of distributional gauge transformations
\mathcal{A}/\mathcal{G}	space of smooth connections modulo smooth gauge transformations
$\overline{\mathcal{A}/\overline{\mathcal{G}}}$	space of distributional connections modulo distributional gauge transformations
$\overline{\mathcal{A}/\mathcal{G}}$	space of distributional gauge equivalence classes of connections
$\overline{\mathcal{A}}^{\mathbb{C}}$	space of distributional complex connections
$\overline{\mathcal{A}/\mathcal{G}}^{\mathbb{C}}$	space of distributional complex gauge equivalence classes of connections
\mathcal{C}	set of semianalytic curves or classical configuration space
$\overline{\mathcal{C}}$	quantum configuration space
\mathcal{P}	set (groupoid) of semianalytic paths or set of punctures
\mathcal{Q}	set (group) of semianalytic closed and basepointed paths
\mathcal{L}	set of tame subgroupoids of \mathcal{P} or general label set
\mathcal{S}	set of tame subgroups of \mathcal{Q} (hoop group) or set of spin-network labels
l	subgroupoid
s	spin-net= spin-network label
$[s]$	(singular) knot-net= diffeomorphism equivalence class of s
Γ_0^{ω}	set of semianalytic, compactly supported graphs
Γ_{σ}^{ω}	set of semianalytic, countably infinite graphs
$\text{Diff}(\sigma)$	group of smooth diffeomorphisms of σ
$\text{Diff}_{\text{sa}}^{\omega}(\sigma)$	group of semianalytic diffeomorphisms of σ
$\text{Diff}_{\text{sa},0}^{\omega}(\sigma)$	group of semianalytic diffeomorphisms of σ connected to the identity
$\text{Diff}_0^{\omega}(\sigma)$	group of analytic diffeomorphisms of σ connected to the identity
$\text{Diff}^{\omega}(\sigma)$	group of analytic diffeomorphisms of σ
φ	(semi-)analytic diffeomorphism
c	semianalytic curve
p	semianalytic path
e	entire semianalytic path (edge)
α	entire semianalytic closed path (hoop) or algebra automorphism

γ	semianalytic graph
v	vertex of a graph
$E(\gamma)$	set of edges of γ
$V(\gamma)$	set of vertexes of γ
$h_p(A) = A(p)$	holonomy of A along p
\prec	abstract partial order
Ω	vector state or symplectic structure or curvature two-form
F	pull-back to σ of 2Ω by a local section
ω	general state on \ast -algebra
\mathfrak{X}, X, Y	measure space or topological space
$L(X, Y), L(X)$	linear (un)bounded operators between X, Y or on X
$B(X, Y), B(X)$	bounded operators between X, Y or on X
$K(X)$	compact operators on X
$B_1(X)$	trace class operators on X
$B_2(X)$	Hilbert–Schmidt operators on X
\mathcal{B}	σ -algebra
μ, ν, ρ	measure
\mathcal{H}	general Hilbert space
Cyl	space of cylindrical functions
\mathcal{D}	dense subspace of \mathcal{H} equipped with a stronger topology
\mathcal{D}'	topological dual of \mathcal{D}
\mathcal{D}^\ast	algebraic dual of \mathcal{D}
$\mathcal{H}^0 = L_2(\overline{\mathcal{A}}, d\mu_0)$	uniform measure L_2 space
\mathcal{H}^\otimes	infinite tensor product extension of \mathcal{H}^0
Cyl_l	restriction of Cyl to functions cylindrical over l
$[\cdot], (\cdot)$	equivalence classes
$\mathfrak{A}, \mathfrak{B}$	abstract (\ast) -algebra or C^\ast -algebra
$\Delta(\mathfrak{A})$	spectrum on Abelian C^\ast -algebra
χ	character (maximal ideal) of unital Banach algebra or group or characteristic function of a set
$\mathfrak{I}, \mathfrak{J}$	ideal in abstract algebra
\mathfrak{P}	classical Poisson \ast -algebra
\mathfrak{G}	automorphism group (of principal fibre bundle)
\mathfrak{D}	Dirac or hypersurface deformation algebra
\mathfrak{M}	Master Constraint algebra
M	Master Constraint