Introduction: Defining quantum gravity

In the first section of this chapter we explain why the problem of quantum gravity cannot be ignored in present-day physics, even though the available accelerator energies lie way beyond the Planck scale. Then we define what a quantum theory of gravity and all interactions is widely expected to achieve and point out the two main directions of research divided into the perturbative and non-perturbative approaches. In the third section we describe these approaches in more detail and finally in the fourth motivate our choice of canonical quantum general relativity as opposed to other approaches.

Why quantum gravity in the twenty-first century?

It is often argued that quantum gravity is not relevant for the physics of this century because in our most powerful accelerator, the LHC to be working in 2007, we obtain energies of the order of a few 10^3 GeV while the energy scale at which quantum gravity is believed to become important is the Planck energy of 10^{19} GeV. While that is true, it is false that nature does not equip us with particles of energies much beyond the TeV scale; we have already observed astrophysical particles with energy of up to 10^{13} GeV, only six orders of magnitude away from the Planck scale. It thus makes sense to erect future particle microscopes not on the surface of the Earth any more, but in its orbit. As we will sketch in this book, even with TeV energy scales one might speculate about quantum gravity effects in the close future with γ-ray burst physics and the GLAST detector. Next, quantum gravity effects in the early universe might have left their fingerprint in the cosmological microwave background radiation (CMBR) and new satellites such as WMAP and PLANCK which have considerably increased the precision of experimental cosmology might reveal those. Notice that these data have already given us new cosmological puzzles recently, namely they have, for the first time, enabled us to reliably measure the energy budget of the universe: about 70% is a so-called dark energy component which could be a positive[1] cosmological constant, about 25% is a dark matter component which is commonly believed to be due to a weakly interacting massive particle (WIMP) (possibly supersymmetric) and only about 5% is made out of baryonic matter. Here 'dark' means

Recent independent observations all indicate that the expansion of the universe is currently accelerating.

that these unknown forms of matter do not radiate, they are invisible. Hence we see that *we only understand 5% of the matter in the universe* and at least as far as dark energy is concerned, quantum gravity could have a lot to do with it. What we want to argue here is that quantum gravity is not at all of academic interest but possibly touches on brand new observational data which point at *new physics beyond the standard model* and are of extreme current interest. See, for example, [18–20] for recent accounts of modern cosmology.

But even apart from these purely experimental considerations, there are good theoretical reasons for studying quantum gravity. To see why, let us summarise our current understanding of the fundamental interactions:

Embarassingly, the only quantum fields that we fully understand to date in four dimensions are *free quantum fields on four-dimensional Minkowski space.* Formulated more provocatively:

In four dimensions we only understand an (infinite) collection of uncoupled harmonic oscillators on Minkowski space!

In order to leave the domain of these rather trivial and unphysical (since non-interacting) quantum field theories, physicists have developed two techniques: perturbation theory and quantum field theory on curved backgrounds. This means the following: with respect to accelerator experiments, the most important processes are scattering amplitudes between particles. One can *formally* write down a unitary operator that accounts for the scattering interaction between particles and which maps between the well-understood free quantum field Hilbert spaces in the far past and future. Famously, by Haag's theorem [21] whenever that operator is really unitary, there is no interaction and if it is not unitary, then it is ill-defined giving rise to the ultraviolet divergences of ordinary QFT. In fact, one can only define the operator perturbatively by writing down the formal power expansion in terms of the generator of the would-be unitary transformation between the free quantum field theory Hilbert spaces. The resulting series is divergent order by order but if the theory is 'renormalisable' then one can make these orders artificially finite by a regularisation and renormalisation procedure with, however, no control on convergence of the resulting series. Despite these drawbacks, this recipe has worked very well so far, at least for the electroweak interaction.

Until now, all we have said applies only to free (or perturbatively interacting) quantum fields on Minkowski spacetime for which the so-called Wightman axioms [21] can be verified. Let us summarise them for the case of a scalar field in $(D+1)$-dimensional Minkowski space:

W1 *Representation*
There exists a unitary and continuous representation $U:\ \mathcal{P} \to \mathcal{B}(\mathcal{H})$ of the **Poincaré group** $\mathcal{P}$ on a Hilbert space $\mathcal{H}$.

W2 *Spectral condition*
The momentum operators $\boldsymbol{P^\mu}$ have spectrum in the forward lightcone: $\boldsymbol{\eta_{\mu\nu} P^\mu P^\nu \leq 0;\ P^0 \geq 0}$.

W3 *Vacuum*
There is a unique **Poincaré** invariant vacuum state $U(\boldsymbol{p})\boldsymbol{\Omega} = \boldsymbol{\Omega}$ for all $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}$.

W4 *Covariance*
Consider the smeared field operator-valued tempered distributions $\phi(f) = \int_{\mathbb{R}^{D+1}} d^{D+1}x \phi(x) f(x)$ where $f \in \mathcal{S}(\boldsymbol{\mathbb{R}^{D+1}})$ is a test function of rapid decrease. Then finite linear combinations of the form $\phi(f_1)\ldots\phi(f_N)\boldsymbol{\Omega}$ lie dense in $\mathcal{H}$ (that is, $\boldsymbol{\Omega}$ is a cyclic vector) and $U(p)\phi(f)U(p)^{-1} = \phi(f \circ p)$ for any $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}$.

W5 *Locality (causality)*
Suppose that the supports (the set of points where a function is different from zero) of f, f' are **spacelike separated** (that is, the points of their supports cannot be connected by a non-spacelike curve) then $[\phi(f), \phi(f')] = 0$.

The most important objects in this list are those that are highlighted in boldface letters: the fixed, non-dynamical Minkowski background metric $\boldsymbol{\eta}$ with its well-defined causal structure, its Poincaré symmetry group $\boldsymbol{\mathcal{P}}$, the associated representation $U(\boldsymbol{p})$ of its elements, the invariant vacuum state $\boldsymbol{\Omega}$ and finally the fixed, non-dynamical topological, differentiable manifold $\boldsymbol{\mathbb{R}^{D+1}}$. Thus the Wightman axioms assume the existence of a non-dynamical, Minkowski background metric which implies that we have a preferred notion of causality (or locality) and its symmetry group, the Poincaré group from which one builds the usual Fock Hilbert spaces of the free fields. We see that the whole structure of the theory is heavily based on the existence of these objects which come with a fixed, non-dynamical background metric on a fixed, non-dynamical topological and differentiable manifold.

For a general background spacetime, things are already under much less control: we still have a notion of causality (locality) but generically no symmetry group any longer and thus there is no obvious generalisation of the Wightman axioms and no natural perturbative Fock Hilbert space any longer. These obstacles can partly be overcome by the methods of algebraic quantum field theory [22] and the so-called microlocal analysis [23–26] (in which the locality axiom is taken care of pointwise rather than globally), which recently have also been employed to develop perturbation theory on arbitrary background spacetimes [27–33] by invoking the mathematically more rigorous implementation of the renormalisation programme developed by Epstein and Glaser in which no divergent expressions ever appear at least order by order (see, e.g., [34]). This way one manages to construct the interacting fields, at least perturbatively, on arbitrary backgrounds.

In order to go beyond a fixed background one can consider 'all backgrounds simultaneously' [35, 36]. Namely, the notion of a local quantum field theory $\mathfrak{A}(M,g)$ (thought of as a unital C^*-algebra for convenience) on a given curved background spacetime (M,g) can be generalised in the following way:[2] given an isometric embedding $\varphi: (M,g) \to (M',g')$ of one spacetime into another, one relates $\mathfrak{A}(M,g)$, $\mathfrak{A}(M',g')$ by asking that there is a $*$-algebraic homomorphism $\alpha_\varphi : \mathfrak{A}(M,g) \to \mathfrak{A}(M',g')$. The homomorphisms α_ψ could for instance just act geometrically by pulling back the fields. More abstractly, what one has then is the category Man whose objects are globally hyperbolic spacetimes (M,g) and whose morphisms are isometric embeddings with unit $\mathbf{1}_{(M,g)} := \mathrm{id}_M$, the identity diffeomorphism. On the other hand, we have the category Alg whose objects are unital C^*-algebras $\mathfrak{A}$ and whose morphisms are injective *-homomorphisms with unit $\mathbf{1}_{\mathfrak{A}} = \mathrm{id}_{\mathfrak{A}}$, the identity element in the algebra. A local quantum field is then a covariant functor $\mathsf{A} : \mathsf{Man} \to \mathsf{Alg}; (M,g) \mapsto \mathfrak{A}(M,g)$, $\varphi \mapsto \alpha_\varphi$ which relates objects and morphisms of Man with those of Alg. The functor is called causal if those quantum field theories $\mathfrak{A}(M_j,g_j)$ for which there exist isometric embeddings $\varphi_j : (M_j,g_j) \to (M,g)$; $j = 1,2$ so that $\varphi_1(M_1)$, $\varphi_2(M_2)$ are spacelike separated with respect to g satisfy the causality axiom $[\alpha_{\varphi_1}(\mathfrak{A}(M_1,g_1)), \alpha_{\varphi_2}(\mathfrak{A}(M_2,g_2))] = \{0\}$. The functor is said to obey the time slice axiom when $\alpha_\varphi(\mathfrak{A}(M,g)) = \mathfrak{A}(M',g'))$ for all isometries $\varphi: (M,g) \to (M',g')$ such that $\varphi(M)$ contains a Cauchy surface for (M',g'). This framework is background-independent because the functor A considers all backgrounds (M,g) simultaneously.

Unfortunately, QFT on curved spacetimes, even stated in this background-independent way, is only an approximation to the real world because it completely neglects the backreaction between matter and geometry which classically is expressed in Einstein's equations. Moreover, it neglects the fact that the gravitational field must be quantised as well, as we will argue below. One can try to rescue the framework of ordinary QFT by studying the quantum excitations around a given classical background metric, possibly generalised in the above background-independent way. However, not only does this result in a non-renormalisable theory without predictive power when treating the gravitational field in the same fashion, it is also unclear whether the procedure leads to (unitarily) equivalent results when using backgrounds which are physically different, such as two Schwarzschild spacetimes with different mass (the corresponding spacetimes are *not isometric*). More seriously, it is expected that especially in extreme astrophysical or cosmological situations (black holes, big bang) the notion of a classical, smooth spacetime **breaks down altogether!** In other words, the fluctuations of the metric operator become deeply quantum and there is no semiclassical notion of a spacetime any more, similarly to the

The following paragraph can be skipped on a first reading, however, the appearing notions are all explained in this book (see, e.g., Definition 6.2.6 and Chapter 29).

energy spectrum of the hydrogen atom far away from the continuum limit. It is precisely here where a full-fledged quantum theory of gravity is needed: we must be able to treat all backgrounds on a common footing, otherwise we will never understand what really happens in a Hawking process when a black hole loses mass due to radiation. Moreover, we need a background-independent theory of GR where the lightcones themselves start fluctuating and hence locality becomes a fuzzy notion. Let us phrase this again, provocatively, as:

The whole framework of ordinary quantum field theory breaks down once we make the gravitational field (and the differentiable manifold) dynamical, once there is no background metric any longer!

Combining these issues, one can say that we have a working understanding of scattering processes between elementary particles in arbitrary spacetimes as long as the backreaction of matter on geometry can be neglected and that the coupling constant between non-gravitational interactions is small enough (with QCD being an important exception) since then the classical Einstein equation, which says that curvature of geometry is proportional to the stress energy of matter, can be approximately solved by neglecting matter altogether. Thus, in this limit, it seems fully sufficient to have only a classical theory of general relativity and perturbative quantum field theory on curved spacetimes.

From a fundamental point of view, however, this state of affairs is unsatisfactory for many reasons among which we have the following:

(i) *Classical geometry – quantum matter inconsistency*
There are two kinds of problem with the idea of keeping geometry classical while matter is quantum:
(i1) *Backreaction*
At a fundamental level, the backreaction of matter on geometry cannot be neglected. Namely, geometry couples to matter through *Einstein's equations*

$$\boxed{\boldsymbol{R_{\mu\nu} - \tfrac{1}{2} R \cdot g_{\mu\nu} = \kappa\, T_{\mu\nu}[g]}}$$

and since matter underlies the rules of quantum mechanics, the right-hand side of this equation, the stress–energy tensor $T_{\mu\nu}[g]$, becomes an operator. One has tried to keep geometry classical while matter is quantum mechanical by replacing $T_{\mu\nu}[g]$ by the Minkowski vacuum Ω_η expectation value $< \Omega_\eta, \hat{T}_{\mu\nu}[\eta]\Omega_\eta >$, but the solution of this equation will give $g \neq \eta$ which one then has to feed back into the definition of the vacuum expectation value, and so on. Notice that the notion of vacuum itself depends on the background metric, so that this is a highly non-trivial iteration process. The resulting iteration does not converge in general [37]. Thus, such a procedure is also inconsistent, whence we *must quantise the gravitational field as well.* This leads to the *quantum Einstein equations*

$$\boxed{\hat{R}_{\mu\nu} - \tfrac{1}{2}\hat{R} \cdot \hat{g}_{\mu\nu} = \kappa\, \hat{T}_{\mu\nu}[\hat{g}]}$$

Of course, this equation is only formal at this point and must be embedded into an appropriate Hilbert space context.

(i2) *UV regime*
There is another piece of evidence for the need to quantise geometry: recall that in perturbative QFT one integrates over virtual particles in higher loop diagrams with arbitrarily large energy. Suppose that such a particle has energy E and momentum $P \approx E/c$ in some rest frame. According to quantum mechanics, such a particle has a lifetime $\tau \approx \hbar/E$ and a spatial extension given by the Compton radius $\lambda \approx \hbar c/E$. According to classical GR, such a lump of energy collapses to a black hole if the Compton radius drops below the Schwarzschild radius $r \approx GE/c^4$, in other words, when the energy exceeds the Planck energy $E_p = \sqrt{\hbar c/G}c^2$. The problem is now not only that in ordinary QFT this general relativistic effect is neglected, but moreover that this effect leads to new processes: according to the Hawking effect, after the lifetime τ the black hole evaporates. However, it evaporates into particles of all possible species. Suppose for instance that the original particle was a neutrino. All that the resulting black hole remembers is its mass and spin. Now while the neutrino only interacts electroweakly according to the standard model, the black hole can produce gluons and quarks, which is impossible within the standard model.

Of course, all of these arguments are only heuristic, however, they reveal that it is problematic to combine classical geometry with quantum matter. They suggest that it is problematic or even inconsistent to resolve spacetime distances below the Planck scale $\ell_p = \sqrt{\hbar c G}/c^2$. It is due to considerations of this kind that one expects that gravity provides a natural UV cutoff for QFT. If that is the case, then it is natural to expect that the quantum spacetime structure reveals a discrete structure at Planck scale. We will see a particular incarnation of this idea in LQG.

(ii) *Inherent classical geometry inconsistency*
Even without quantum theory at all Einstein's field equations predict spacetime singularities (black holes, big bang singularities, etc.) at which the equations become meaningless. In a truly fundamental theory, there is no room for such breakdowns and it is suspected by many that the theory cures itself upon quantisation in analogy to the hydrogen atom whose stability is classically a miracle (the electron should fall into the nucleus after a finite

time lapse due to emission of Bremsstrahlung) but is easily explained by quantum theory which bounds the electron's energy from below.

(iii) *Inherent quantum matter inconsistency*
As outlined above, perturbative quantum field theory on curved spacetimes is itself also ill-defined due to its UV (short distance) singularities which can be cured only with an ad hoc recipe order by order which lacks a fundamental explanation; moreover, the perturbation series is usually divergent. Besides that, the corresponding infinite vacuum energies being usually neglected in such a procedure contribute to the cosmological constant and should have a large gravitational backreaction effect. That such energy subtractions are quite significant is maybe best demonstrated by the Casimir effect. Now, since general relativity possesses a fundamental length scale, the Planck length $\ell_p \approx 10^{-33}$ cm, it has been argued ever since that gravitation plus matter should give a finite quantum theory since gravitation provides the necessary, built-in, short distance cutoff.

(iv) *Cosmological constant problem*
However, that cutoff cannot work naively: consider for simplicity a free massless scalar field on Minkowski space. The difference between the Hamiltonian and its normal ordered version is given by the divergent expression

$$\hat{H}- : \hat{H} := \hbar \int d^3x [\sqrt{-\Delta}\delta(x,y)]_{y=x} = \hbar \int d^3x \int d^3k \; |k|$$

where Δ is the flat space Laplacian. If we assume a naive momentum cutoff due to quantum gravity at $|k| \le 1/\ell_P$ the divergent momentum integral becomes proportional to ℓ_P^{-4}. Comparing this with the cosmological constant Hamiltonian $\frac{\Lambda}{G} \int d^3x \sqrt{\det(q)}$ where Λ is the cosmological constant, G is Newton's constant and q is the spatial metric (which is flat on Minkowski space) then we conclude that $\Lambda \ell_P^2 \approx 1$ where $\hbar G = \ell_P^2$ was used. However, experimentally we find $\Lambda \ell_P^2 \approx 10^{-120}$. Thus the cosmological constant is unnaturally small and presents the worst fine-tuning problem ever encountered in physics. Notice that the cosmological constant is a possible candidate for dark energy.

(v) *Perturbative quantum gravity inconsistency*
Given the fact that perturbation theory works reasonably well if the coupling constant is small for the non-gravitational interactions on a background metric it is natural to try whether the methods of quantum field theory on curved spacetime work as well for the gravitational field. Roughly, the procedure is to write the dynamical metric tensor as $g = \eta + h$ where η is the Minkowski metric and h is the deviation of g from it (the graviton) and then to expand the Lagrangian as an infinite power series in h. One arrives at a formal, infinite series with finite radius of convergence which becomes meaningless if the fluctuations are large. Although the naive power counting argument implies that general relativity so defined is a non-renormalisable

theory, it was hoped that due to cancellations of divergences the perturbation theory could actually be finite. However, that this hope was unjustified was shown in [38, 39] where calculations demonstrated the appearance of divergences at the two-loop level, which suggests that at every order of perturbation theory one must introduce new coupling constants which the classical theory did not know about and one loses predictability.

It is well known that the (locally) supersymmetric extension of a given non-supersymmetric field theory usually improves the ultraviolet convergence of the resulting theory as compared with the original one due to fermionic cancellations [40]. It was therefore natural to hope that quantised supergravity might be finite. However, in [41] a serious argument against the expected cancellation of perturbative divergences was raised and recently even the again popular (due to its M-theory context) most supersymmetric 11D 'last hope' supergravity theory was shown not to have the magical cancellation property [42–44].

Summarising, although a definite proof is still missing up to date (mainly due to the highly complicated algebraic structure of the Feynman rules for quantised supergravity) it is today widely believed that perturbative quantum field theory approaches to quantum gravity are meaningless.

The upshot of these considerations is that our understanding of quantum field theory and therefore fundamental physics is quite limited unless one quantises the gravitational field as well. Being very sharply critical one could say:

The current situation in fundamental physics can be compared with the one at the end of the nineteenth century: while one had a successful theory of electromagnetism, one could not explain the stability of atoms. One did not need to worry about this from a practical point of view since atomic length scales could not be resolved at that time but from a fundamental point of view, Maxwell's theory was incomplete. The discovery of the mechanism for this stability, quantum mechanics, revolutionised not only physics. Similarly, today we still have no thorough understanding for the stability of nature in the sense discussed above and it is similarly expected that the more complete theory of quantum gravity will radically change our view of the world. That is, considering the metric as a quantum operator will bring us beyond standard model physics even without the discovery of new forces, particles or extra dimensions.

The role of background independence

The twentieth century has dramatically changed our understanding of nature: it revealed that physics is based on two profound principles, quantum mechanics and general relativity. Both principles revolutionise two pivotal structures of

Newtonian physics. First, the determinism of Newton's equations of motion evaporates at a fundamental level, rather dynamics is reigned by probabilities underlying the Heisenberg uncertainty obstruction. Second, the notion of absolute time and space has to be corrected; space and time and distances between points of the spacetime manifold, that is, the metric, become themselves dynamical, geometry is no longer just an observer. The usual Minkowski metric ceases to be a distinguished, externally prescribed, background structure. Rather, the laws of physics are *background-independent*, mathematically expressed by the classical Einstein equations which are *generally (or four-diffeomorphism) covariant*. As we have argued, it is this new element of *background independence* brought in with Einstein's theory of gravity which completely changes our present understanding of quantum field theory.

A satisfactory physical theory must combine both of these fundamental principles, quantum mechanics and general relativity, in a consistent way and will be called 'Quantum Gravity'. However, the quantisation of the gravitational field has turned out to be one of the most challenging unsolved problems in theoretical and mathematical physics. Although numerous proposals towards a quantisation have been made since the birth of general relativity and quantum theory, none of them can be called successful so far. This is in sharp contrast to what we see with respect to the other three interactions whose description has culminated in the so-called standard model of matter, in particular, the spectacular success of perturbative quantum electrodynamics whose theoretical predictions could be verified *to all digits within the experimental error bars* until today.

Today we do not have a theory of quantum gravity, what we have is:

1. The Standard Model, a quantum theory of the non-gravitational interactions (electromagnetic, weak and strong) or *matter* which, however, completely ignores General Relativity.
2. Classical General Relativity or *geometry*, which is a background-independent theory of all interactions but completely ignores quantum mechanics.

What is so special about the gravitational force that it has persisted in its quantisation for about 70 years already? As outlined in the previous section, *the answer is simply that today we only know how to do QFT on fixed background metrics*. The whole formalism of ordinary QFT relies heavily on this background structure and collapses to nothing when it is missing. It is already much more difficult to formulate a QFT on a non-Minkowski (curved) background but it seems to become a completely hopeless task when the metric is a dynamical, even fluctuating quantum field itself. This underlines once more the source of our current problem of quantising gravity: *we have to learn how to do QFT on a differential manifold (or something even more rudimentary, not even relying on a fixed topological, differentiable manifold) rather than a spacetime.*

In order to proceed, today a high-energy physicist has the choice between the following two, extreme approaches. Either the *particle physicist's*, who prefers to take over the well-established mathematical machinery from QFT on a background at the price of dropping background independence altogether to begin with and then tries to find the true background-independent theory by summing the perturbation series (summing over all possible backgrounds). Or the *quantum geometer's*, who believes that background independence lies at the heart of the solution to the problem and pays the price to have to invent mathematical tools that go beyond the framework of ordinary QFT right from the beginning. Both approaches try to unravel the truly deep features that are unique to Einstein's theory associated with background independence from different ends.

The particle physicist's language is perturbation theory, that is, one writes the quantum metric operator as a sum consisting of a background piece and a perturbation piece around it, the graviton, thus obtaining a graviton QFT on a Minkowski background. *We see that perturbation theory, by its very definition, breaks background independence and diffeomorphism invariance at every finite order of perturbation theory.* Thus one can restore background independence only by summing up the entire perturbation series, which is of course not easy. Not surprisingly, as already mentioned, since $\hbar\kappa = \ell_p^2$ has negative mass dimension in Planck units, applying this programme to Einstein's theory itself results in a mathematical disaster, a so-called non-renormalisable theory without any predictive power. In order to employ perturbation theory, it seems that one has to go to string theory which, however, requires the introduction of new additional structures that Einstein's classical theory did not know about: supersymmetry, extra dimensions and an infinite tower of new and very heavy particles next to the graviton. This is a fascinating but extremely drastic modification of general relativity and one must be careful not to be in conflict with phenomenology as superparticles, Kaluza Klein modes from the dimensional reduction and those heavy particles have not been observed until today. On the other hand, string theory has a good chance to be a unified theory of the perturbative aspects of all interactions in the sense that all interactions follow from a common object, the string, thereby explaining the particle content of the world.

The quantum geometer's language is a non-perturbative one, keeping background independence as a guiding principle at every stage of the construction of the theory, resulting in mathematical structures drastically different from the ones of ordinary QFT on a background metric. One takes Einstein's theory absolutely seriously, uses only the principles of General Relativity and quantum mechanics and lets the theory build itself, driven by mathematical consistency. Any theory meeting these standards will be called *Quantum General Relativity (QGR)*. Since QGR does not modify the matter content of the known interactions, QGR is therefore not in conflict with phenomenology but also it does not obviously explain the particle content of the world. However, it tries to unify all interactions in a different sense: all interactions must transform under a common gauge group, the four-dimensional diffeomorphism group which on the other hand is almost completely broken in perturbative approaches.