Introduction

MOTIVATION

The idea for this book arose when the authors discovered, working together on a particular problem in elastic contact mechanics, that they were making extensive and repeated use of MATHEMATICA™ as a powerful, convenient, and versatile tool. Critically, the usefulness of this tool was not limited to its ability to compute and display complex two- and three-dimensional fields, but rather it helped in understanding the relationships between different vector and tensor quantities and the way these quantities transformed with changes of coordinate systems, orientation of surfaces, and representation.

We could still remember our own experiences of learning about classical elasticity and tensor analysis, in which grasping the complex nature of the objects being manipulated was only part of the challenge, the other part being the ability to carry out rather long, laborious, and therefore error-prone algebraic manipulations.

It was then natural to ask the question: Would it be possible to develop a set of algebraic instruments, within MATHEMATICA, that would carry out these laborious manipulations in a way that was transparent, invariant of the coordinate system, and error-free? We started the project by reviewing the existing MATHEMATICA packages, in particular the `VectorAnalysis` package, to assess what tools had been already developed by others before us, and what additions and modifications would be required to enable the manipulation of second-rank tensor field quantities, which are of central importance in classical elasticity. In this book we present our readers with the result of our effort, in the form of MATHEMATICA packages, notebooks, and worked examples.

In the course of building up this body of methods and solutions, we were forced to review much of the well-established body of classical elasticity, looking for areas of application where our instrumentarium would be most effective. After a while it became apparently necessary for us to include this review in the text, in order to preserve the logic and consistency of approach and to achieve a level of completeness – although we did not aim to reach every region of the vast domain of continuum mechanics, or elasticity in particular.

This book is intended as a text and reference for those wishing to realise more fully the benefit of studying and using classical elasticity. The approaches presented here are not aimed at replacing various other computational techniques that have become successful and widespread in modern engineering practice. Finite element methods, in particular, through decades of application and development, have acquired tremendous versatility and the ability to deliver numerical solutions of complex problems. However, the power of analytical treatments possible within the framework of elasticity should not be underestimated: true understanding of physical systems often consists of the ability to identify the relationships and interdependencies between different quantities, and nothing serves this objective more elegantly and efficiently than concise analytical solutions.

It is our hope that any readers who have previous experience of courses in engineering mechanics and strength of materials will find something useful for themselves in this book. This might be just a practical tool, such as a symbolic manipulation module; or it might be an explanation that helps readers to make sense of a more or less sophisticated concept in elasticity theory, or in the broader context of continuum mechanics. In particular, we sought to use consistently, insofar as it was possible, the invariant form of operations with tensor fields. It is of course true that for practical purposes the results always need to be expressed in some specific coordinate system, to make them understandable to computer algebra systems and humans. Natural phenomena, however, do not require coordinate systems to happen – in fact, some of the most successful theories in the natural sciences are built on the basis of invariance with respect to transformations of spatial and temporal coordinates. The great benefit of the symbolic manipulation ability of MATHEMATICA is that it allows the (sometimes heavy) machinery of tensor manipulation in index notation to be hidden from the user. It is indeed our hope that providing readers with coordinate-invariant analytical instruments will allow them to concentrate on the intriguing underlying natural relationships that are the reason many people choose to study this subject in the first place.

Many books exist that are devoted to similar topics, and many of them are remarkably good. Some of them show readers in detail how important results in elasticity are derived, often frightening away beginners with lengthy derivations and numerous indices. Others select some of the most elegant solutions that can be obtained in a surprisingly concise way, if the right path to the answer is judiciously chosen, usually on the basis of many years of practice in algebraic manipulation. This work is unique in that it attempts to place the focus firmly on the analysis of the mechanics of deformation in terms of tensor fields, but to take away the fear of 'long lines,' freeing the reader to explore, verify, visualise, and compute.

As in any classical subject (and there are not many fields in hard natural science more classically established than elasticity), a great body of knowledge has been accumulated over decades and centuries of research. Detailed description of all of these areas could fill many volumes. Topics covered in this book were selected because they represent the common core of concepts and methods that will be useful to any practitioner, whether on the research or application side of the subject. They also lend themselves well to being implemented in the form of symbolic manipulation packages and illustrate key principles that could be applied elsewhere within the broader subject. We made a deliberate effort to make this book rather concise, aiming to illustrate an approach that can be successfully applied also to numerous other examples found in the excellent literature on the subject.

The authors' experience is primarily of teaching continuum mechanics and elasticity to European students in France and the United Kingdom. Some of the material included in this book was used to teach advanced mechanics and stress analysis courses. However, it is also the authors' belief that, in the context of the U.S. graduate teaching system, the scope covered in this work would be particularly appropriate for a one-semester course at the graduate level in departments of engineering mechanics, engineering science, and mechanical, aerospace, and civil engineering. It will equip the listeners with valuable analytical skills applicable in many contexts of applied research and advanced industrial development work.

The subject of the book is of particular interest to the authors because both of them have been involved, for a number of years, in the capacity of researchers, graduate supervisors, research project leaders, and consultants, in the application of classical methods of continuum mechanics to modern engineering problems in the aerospace and automotive industry, power generation, manufacturing optimisation and process modelling, systems design, and structural integrity assessment, etc.

Classical elasticity is one of the oldest and most complete theories in modern science. Its development was driven by engineering demands in both civil and military construction and manufacture and required the invention and refinement of analytical tools that made crucial contributions to the broader subject of applied mathematics.

In an old and thoroughly researched subject such as elasticity, why does one need yet another textbook? Elasticity theory has not experienced the kind of revolution brought about by quantum theory in physics or the discovery of the gene in biology. Development of elasticity theory largely followed the paradigm established by Cauchy and Kelvin, Lagrange and Love, without significant revisions. Certainly, one ought not to overlook the advent of powerful computational techniques such as the boundary element method and the finite element method. Yet these techniques are entirely numerical in their nature and cannot be used directly to establish fundamental analytical relations between various problem parameters.

For the first time in perhaps over 200 years, the practice of performing analytical manipulations in elasticity is changing from the pen and paper paradigm to something entirely different: analytical elasticity by computer.

The origins of elasticity are often traced to Hooke's statement of elasticity in 1679 in the form of the anagram CEIIINSSSTTUVO containing the coded Latin message 'ut tensio, sic vis,' or 'as the extension, so the force.' Development of elasticity theory required generalisation of the concepts of extension or deformation and of stress to three dimensions. The necessity of describing elastic fields promoted the development of vector analysis, matrix methods, and particularly tensor calculus. The modern notation used in tensor calculus is largely due to Ricci and Levi-Civita, but the term 'tensor' itself was first introduced by Voigt in 1903, possibly in reference to Hooke's 'tensio.'

The subject of tensor analysis is thus particularly closely related to elasticity theory. In this book we devote particular attention to the manipulation of second rank tensors in arbitrary orthogonal curvilinear coordinate systems to derive elastic solutions. Differential operations with second rank tensors are considered in detail in an appendix. Most importantly from the practical viewpoint, convenient tools for tensor manipulation are written as modules or commends and organized in the form of a MATHEMATICA package supplied with this book.

The theory of potential is another branch of mathematics that stands in a close symbiotic relationship with elasticity theory, in that it both was driven by and benefited from the search for solutions of practical elasticity problems. We devote particular attention to potential representations of elastic fields, in terms of both stress and displacement functions.

Fundamental theorems of elasticity are indispensible tools needed to establish uniqueness of solutions and also to develop the techniques for finding approximate solutions. These are presented in a concise form, and their use is illustrated using Mathematica examples. Particular attention is given to the development of approximate solution techniques based on rigorous variational arguments.

Appendices contain some reference information, which we hope readers will find useful, on tensor calculus and Mathematica commands employed throughout the text.

WHAT WILL AND WILL NOT BE FOUND IN THIS BOOK

The particular emphasis in this text is placed on developing a Mathematica instrumentarium for manipulating vector and tensor fields in invariant form, but also allowing the user to inspect and dissect the expressions for tensor components in explicit, coordinate-system-specific forms. To this end, at relevant points in the presentation, the appropriate modules are constructed. This includes the definition of differential operators (**`Grad, Div, Curl, Laplacian, Biharmonic, Inc`**) applicable to scalars, vectors, and tensors. Importantly, in the case of tensor fields, definitions of right (post-) and left (pre-) forms of the **`Grad`** operator are made available. Analysis of biharmonic functions is addressed in some detail, and tools for the reduction of differential operators in arbitrary orthogonal curvilinear coordinate systems are provided to help the reader reveal and appreciate their nature. The modules **`IntegrateGrad`** and **`IntegrateStrain`** have particular significance in the context of linear elastic theory and are explained in some detail, together with their connection with the Saint Venant strain compatibility conditions. All packages, example notebooks, and solutions to exercises can be downloaded freely from the publisher's web site at www.cambridge.org/9780521842013

The development of Mathematica tools happens against the backdrop of the presentation of the classical linear elastic theory. To keep the presentation concise, some care was taken to select the topics included in this treatment.

Chapter 1 is devoted to the kinematics of motion and serves as a vehicle for introducing the concept of deformation as a transformation map, leading naturally to the concept of deformation gradient and its polar decomposition into rotation and translation. The definition of strain then follows, and particular attention is focused on the concept of small strain. The procedure for reconstituting the displacement field from a given distribution of small strains is constructed based on rigorous arguments and implemented in the form of an efficient Mathematica module. In the process of developing this constructive approach, the conditions for small strain integrability are identified (also known as the Saint Venant strain compatibility conditions).

The significance of some differential operators applied to tensor fields becomes immediately apparent from the analysis of Chapter 1. In particular, the second-order incompatibility operator, inc , is introduced, allowing the Saint Venant condition for compatibility of small strain $\boldsymbol{\epsilon}$ to be written concisely:

$$\operatorname{inc} \boldsymbol{\varepsilon} = \mathbf{0}.$$

This operator has particular significance in the theory of elasticity, and further attention is devoted to it in subsequent chapters, as well as to its relationship with the laplacian and biharmonic operators.

Chapter 2 is devoted to the analysis of forces. Particular attention is given to elastostatics, that is, the study of stresses and the conditions of their equilibrium. We show how the principle of virtual power offers a rational starting point for the analysis of equilibria of continua. The concept of stress appears naturally in this approach as dual to small strain in a continuum solid. Furthermore, the equations of stress equilibrium, together with the traction boundary conditions, follow from this variational formulation in the most convenient invariant form. An interesting aside here is the discussion of the expressions for virtual power arising within different kinematical descriptions of deformation (e.g., inviscid fluid, beams under bending) and the modifications of the concept of stress that are appropriate for these cases.

The classical stress definition according to Cauchy is also presented, and its equivalence to the definition arising from the principle of virtual power is noted. The Cauchy–Poisson theorem then establishes the form of equilibrium equations and traction boundary conditions. (Discussion of the index form of equilibrium equations and boundary conditions that is specific to coordinate systems is addressed by demonstration in the exercises at the end of this chapter.) Some elementary stress states are considered in detail.

Having established the fact that equlibrium stress states in continuum solids in the absence of body forces are represented by divergence-free tensors, we address the question of efficient representation of such tensor fields. The Beltrami potential representation is introduced, in which the operator inc once again makes its appearance. Donati's theorem is then quoted, which establishes a certain duality between the conditions of stress equilibrium and strain compatibility.

Chapter 3 is devoted to the discussion of general anisotropic elasticity tensors. Important properties of elastic tensors are introduced, and the relationships between tensor and matrix representations are rigorously considered, together with efficient MATHEMATICA implementations of conversion between different forms. Next, classes of material elastic sysmmetry are considered, and the implications for the form of elastic stiffness matrices are clarified. Elastic isotropy is discussed in detail as a particularly important case that is treated in more detail in subsequent chapters.

MATHEMATICA tools for displaying elastic symmetry planes are presented, along with ways of visualising the results of extension experiments on anisotropic materials.

The methods of solution of elasticity problems for anisotropic materials are not considered in the present treatment, as the authors felt that this important subject deserved special treatment.

Modifications and perturbations to the liniear elastic theory are briefly discussed, including thermal strain effects and residual stresses. The chapter is concluded with a brief discussion of the limitations of the linear elastic theory and the formulation of Tresca and von Mises yield criteria.

Chapter 4 is devoted to the formulation of the complete problem of elasticity and the discussion of general theorems and principles. First, the formulation of a well-posed, or regular problem of thermoelasticity is introduced. Next, the displacement formulation (Navier equation) and the stress formulation (Beltrami–Michell equations) are introduced. As a demonstration of the application of elasticity problem formulation, the problem of the spherical vessel is solved directly by considering the radial displacement field in the spherical coordinate system, computing strains and stresses, and satisfying the equilibrium and boundary conditions.

Next, the principle of superposition is introduced, followed by the virtual work theorem. This allows the nature and the conditions for the uniqueness of elastic solution to be established. This is followed by the proof of existence of the strain energy potential and the complementary energy potential and of reciprocity theorems. Saint Venant torsion is next considered in detail with the help of Mathematica implementation, serving as the vehicle for the introduction of the more general Saint Venant principle. The counterexample due to Hoff is given as an illustration, and a rigorous formulation of the principle, following von Mises and Sternberg, is given.

Chapter 5 is devoted to the solution of elastic problems using the stress function approach. The Beltrami potential introduced previously provides a convenient representation of self-equilibrated stress fields. The Airy stress function corresponds to a particular case of this representation and is of special importance in the context of plane elasticity due to its simplicity, and for historical reasons. Particular care is therefore taken to introduce this approach and to discuss the precise nature of strain compatibility conditions that must be imposed in this formulation to complement stress equilibrium. This allows the elucidation of the strain incompatibility that arises in the plane stress approximation. In passing, an important issue of verifying the biharmonic property of expressions in an arbitrary coordinate system is addressed symbolically through the analysis of reducibility of differential operators. It is then demonstrated how strain compatibility in plane stress can be enforced through the introduction of a corrective term. Plane strain is also considered, and the simple relationship with plane stress is pointed out.

The properties of Airy stress functions in cylindrical polar coordinates are addressed next. The general form of biharmonic functions of two coordinates, due to Goursat, serves as the basis for obtaining various forms of Airy stress functions as suitable candidate solutions of the plane elasticity problem. The Michell solution, although originally incomplete and amplified with additional terms by various contributors, is introduced and discussed due to its historical importance. Furthermore, it allows the identification of some important fundamental solutions that serve as *nuclei of strain* within the elasticity theory. In this way the solutions for disclination, dislocation, and other associated problems are analysed.

The Airy stress function solution is derived next for a concentrated force applied at the apex of an infinitely extended wedge. This important solution serves to introduce the Flamant solution for the concentrated force at the surface of an elastic half-plane. The combination of the appropriate wedge solution with the dislocation solution allows the Kelvin solution for a concentrated force acting in an infinite elastic plane to be derived by enforcing displacement continuity. The derivation makes use of the strain integration procedure presented earlier.

Williams eigenfunction analysis of the stress state in an elastic wedge under homogeneous loading is presented next. On the basis of this solution, the elastic stress fields can be found around the tip of a sharp crack subjected either to opening or to shear mode loading. Finally, two further important problems are treated, namely the Kirsch problem of remote loading of a circular hole in an infinite plate and the Inglis problem of remote loading of an elliptical hole in an infinite plate.

Chapter 6 is devoted to the introduction and use of the method of displacement potential. First, the harmonic scalar and vector Papkovich–Neuber potentials are introduced and the representations of simple deformation states in terms of these potentials are found. Next, the fundamental solution of three-dimensional elasticity is derived, the Kelvin solution for a force concentrated at a point within an infinitely extended isotropic elastic solid. The Kelvin solution serves as the basis for deriving solutions for force doublets, or dipoles, with or without moment, and also for centres of dilatation and rotation. These are further examples of *strain nuclei*, already introduced earlier in the context of plane elastic problems.

Solutions presented next are for the Boussinesq and Cerruti problems about concentrated forces applied normally or tangentially to the surface of an elastic half-space. The solution for a concentrated force applied at the tip of an elastic cone is given next. General solutions in spherical and cylindrical coordinates are discussed, and the use of spherical harmonics illustrated. The Galerkin vector is introduced as an equivalent displacement potential formulation, and Love strain function presented as a particular case. The chapter is concluded with a brief note on the integral transform methods and contact problems.

Chapter 7 deals with the subject of energy principles and variational formulations, which are of particular importance for many applications, because they provide the basis for most numerical methods of approximate solutions for problems in continuum solid mechanics. Using strain energy and complementary energy potentials introduced earlier, a suite of extremum theorems is introduced. On this basis approximate solutions (bounds) in the theory of elasticity are introduced, using the notions of kinematically and statically admissible fields. The problem of the compression of a cylinder between rigid platens provides an example of application of the method.

Next, extremal properties of free vibrations and approximate spectra are considered. Analysis of vibration of a cantilever beam serves as an example.

Appendices contain some background information on linear differential operators, particularly in application to tensor fields studied with respect to general orthogonal curvilinear coordinate systems. Also explained is the implementation of these operators within the `Tensor2Analysis` package. Some important MATHEMATICA constructs used in the text, such as the `IntegrateGrad` module, are also explained, along with other MATHEMATICA tricks and utilities developed by the authors for the visualisation of results.

This book does not dwell in any detail on many important problems in elasticity and continuum solid mechanics. Anisotropic elasticity problems are not addressed here in any detail, nor are the complex variable methods in plane elasticity. Contact mechanics forms another large section of elasticity that is not treated here. Elastic waves, dispersion, and interaction with boundaries are not addressed in this text, again due to the fact that the authors thought it impossible to give a fair exposition of this subject within the limited space available.

It is the authors' hope, however, that many of the methods and approaches developed and presented in this book will provide the reader with transferable techniques that can be applied to many other interesting and complex problems in continuum mechanics. To help achieve this purpose, the book contains over 60 exercises that are most efficiently solved using MATHEMATICA tools developed in the corresponding chapters. Many of these exercises are not original, and, whenever possible, explicit reference is made to the source. The authors' hope is, however, that in solving all of these exercises readers will be able to appreciate the advantages offered by symbolic manipulation.

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Andrei Constantinescu and Alexander Korsunsky
Excerpt
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1 Kinematics: displacements and strains

OUTLINE

This chapter is devoted to the introduction of the fundamental concepts used to describe continuum deformation. This is probably most naturally done using examples from fluid dynamics, by considering the description of particle motion either with reference to the initial particle positions, or with reference to the current (actual) configuration. The relationship between the two approaches is illustrated using examples, and further illustrations are provided in the exercises at the end of the chapter. Some methods of flow visualisation (streamlines and streaklines) are described and are illustrated using simple examples. The concepts are then clarified further using the example of inviscid potential flow.

Placing the focus on the description of deformation, the fundamental concept of deformation gradient is introduced. The polar decomposition theorem is used to separate deformation into rotation and stretch using appropriate tensor forms, with particular attention being devoted to the analysis of the stretch tensor and the principal stretches, using pure shear as an illustrative example. Trigonometric representation of stretch and rotation is discussed briefly.

Discussion is further specialised to the consideration of small strains. Analysis of integrability of strain fields then leads to the identification of the invariant form of compatibility conditions. This subject is important for many applications within elastic theory and is therefore dwelt on in some detail. Strain integration is implemented as a generic module in MATHEMATICA, allowing displacement field reconstruction within any properly defined orthogonal curvilinear system.

1.1 PARTICLE MOTION: TRAJECTORIES AND STREAMLINES

Lagrangian description

Let us suppose that the material body under observation occupies the domain $\Omega \in \mathbb{R}^3$ in a reference configuration $\mathcal{C}$. Each material point is identified by its spatial position $\boldsymbol{X}$ in the reference configuration.

Let us assume that the motion of a particle is described by a function

$$\boldsymbol{x} = \mathcal{F}(\boldsymbol{X}, t) \tag{1.1}$$

which maps each point $\boldsymbol{X}$ of the reference configuration onto its position $\boldsymbol{x}$ at time t.

The mapping $\mathcal{F}$ is therefore defined,

$$\mathcal{F} : \Omega \times [0, T] \longrightarrow \mathbb{R}^3,$$

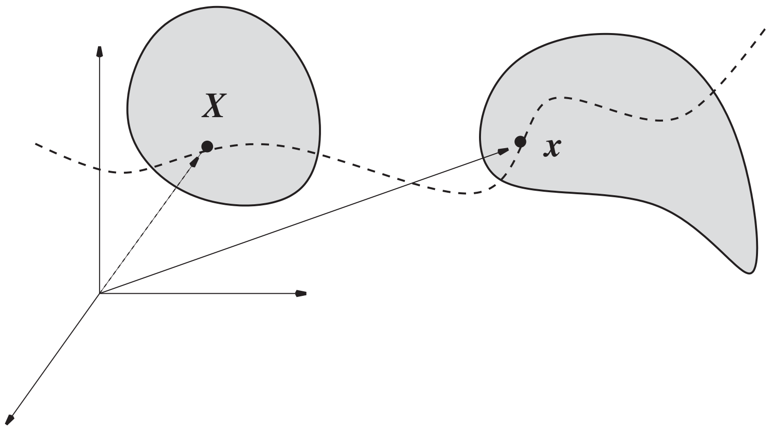

Figure 1.1. The initial and the actual configuration of a body and the position of a particle on its path.

where Ω denotes the initial configuration of the body. The domain $\Omega_t = \boldsymbol{\mathcal{F}}(\Omega, t)$ is referred to as the actual configuration at time t.

This description of motion is referred to as the *Lagrangian* description.

We shall assume that matter is neither created nor removed, that noninterpenetrability of particles is respected, and that the continuity of material orientation is conserved during motion.

These assumptions imply that there exists a one-to-one relation between material particles and points $\boldsymbol{X}$ in the reference configuration, as well as between the initial and actual positions of particles $\boldsymbol{X}$ and $\boldsymbol{x}$, respectively.

Particle path

The trajectory of a given particle in the fixed laboratory frame is the curve that is also referred to as the *particle path* (see Figure 1.1). The particle path is the geometrical locus of the points occupied by the material particle at different times during deformation and can be mathematically expressed as the following set:

$$\mathcal{P}(\boldsymbol{X}) = \{\boldsymbol{\mathcal{F}}(\boldsymbol{X}, t) | t \in [0, T]\}. \tag{1.2}$$

Consider as an example the particle paths of points on a rigid 'railway' wheel that is rolling without slipping along a surface represented by a straight horizontal line.

In order to illustrate particle paths in MATHEMATICA, first define the transformation $\mathcal{F}$ that at time t is given by the superposition of translation of the wheel centre by the distance $\boldsymbol{v}t$ and rotation of the wheel around its centre by the angle ωt. This is done by introducing vector positions of the wheel centre `cent`, the particle point a and velocity vector v, and the rotation matrix `rot`.

```
a = {a1, a2, a3}; v = {v1, 0, 0};

rot[phi_] :=
```

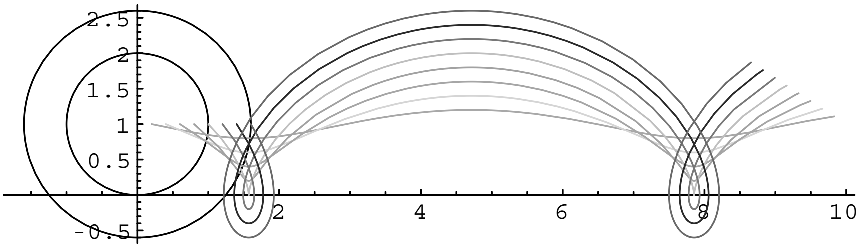

Figure 1.2. Trajectories of points on a rigid 'railway' wheel rolling along a horizontal surface without slipping.

```
   {{ Cos[phi], Sin[phi], 0},
    {-Sin[phi], Cos[phi], 0},
    {        0,        0, 1}}

F[a_, t_]:= v t + cent + rot[omega t].(a - center)
F[a, t]
```

The condition of rolling without slipping is ensured by the fact that the total velocity of the point in instantaneous contact with the surface is equal to zero, due to the fact that the contributions to this velocity from the translation and rotation parts of the motion are equal and opposite; that is,

$$v_1 = \omega R.$$

The points selected for particle path tracking are obtained as a double-indexed list using **Table**. **Flatten** transforms the double-indexed list into a single-indexed list.

The **wheel** and **wheel1** represent the 'railway' wheel with an outsized 'tyre' that is allowed to pass below the surface. The **traject** set of particle paths is obtained using the standard **ParametricPlot** command. The form of the command represents the application (**Map**) of the **ParametricPlot** command to all initial **points**. The **Drop** command eliminates the third coordinate for two-dimensional plotting.

All trajectories and the wheel and tyre are displayed in Figure 1.2 using **Show**. The particle paths can be recognised as *cycloids*. Classical implicit equations for these curves can be obtained after some additional manipulations.

```
cent = {0, R, 0};
R = 1; omega = 1; v1 = R omega;

points  = Flatten[
 Table[
  center + r {Cos[alpha], Sin[alpha], 0},
    {r, 0.2, 1.6, 0.2}, {alpha, 0, 0}], 1]

wheel = ParametricPlot[
   {R Cos[theta], 1 + R Sin[theta]},
```