OUTER CIRCLES

We live in a three-dimensional space; what sort of space is it? Can we build it from simple geometric objects? The answers to such questions have been found in the last 30 years, and *Outer Circles* describes the basic mathematics needed for those answers as well as making clear the grand design of the subject of hyperbolic manifolds as a whole.

The purpose of *Outer Circles* is to provide an account of the contemporary theory, accessible to those with minimal formal background in topology, hyperbolic geometry, and complex analysis. The text explains what is needed, and provides the expertise to use the primary tools to arrive at a thorough understanding of the big picture. This picture is further filled out by numerous exercises and expositions at the ends of the chapters and is complemented by a profusion of high quality illustrations. There is an extensive bibliography for further study.

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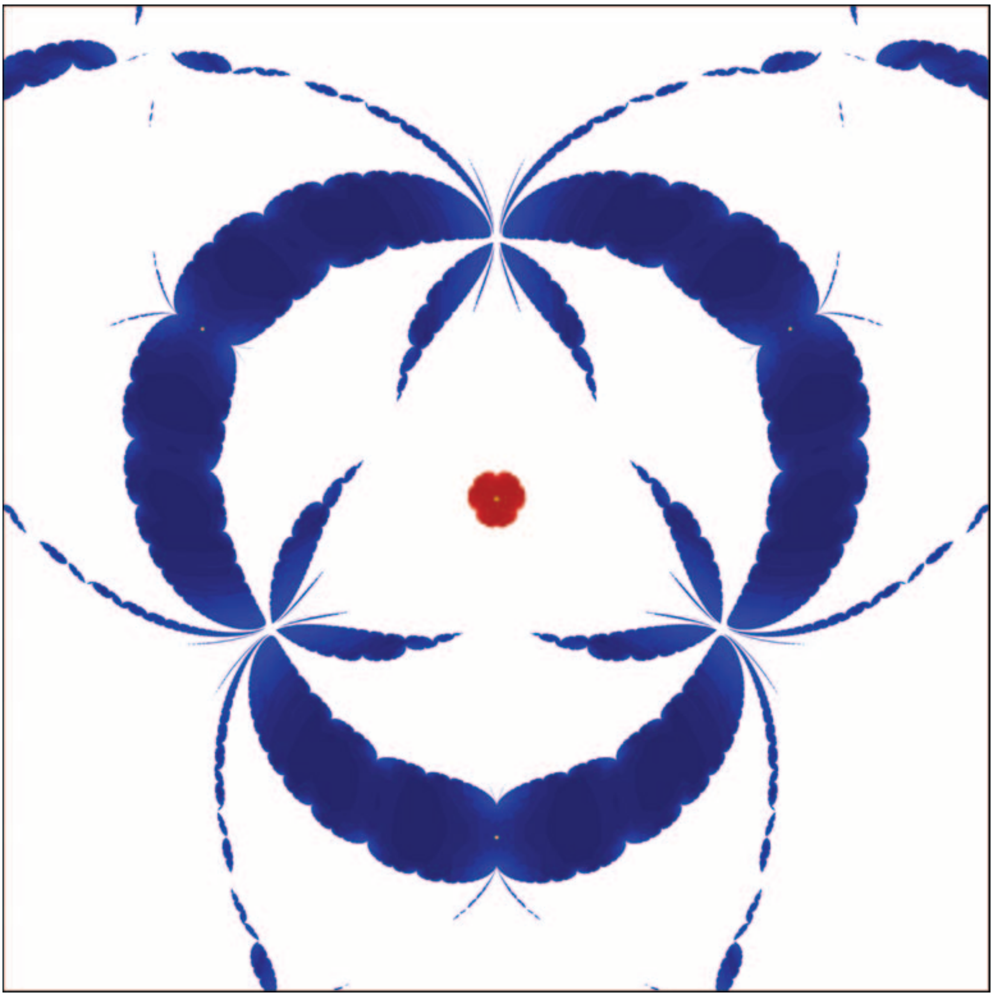

The discreteness locus in the extended Bers slice of the hexagonal once-punctured torus (see Exercise 6-8). The Bers slice—the red central object—is surrounded by other islands of discontinuity, in blue. The inward pointing cusps on the Bers slice boundary represent geometrically finite groups and the same is presumably true for the other components. The yellow dots are the fuchsian centers of the components. Only a small number of islands are shown because of theoretical and computational limitations.

The computation and image were made by David Dumas of Brown University; his web site contains many beautiful related images.

OUTER CIRCLES

An Introduction to Hyperbolic 3-Manifolds

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A. Marden
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More information

To Dorothy

צו מיין פרוי דבורה, די אמת'דיקע אשת-חיל.

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List of Illustrations

Credits: CM, Curt McMullen; CS, Caroline Series; DD, David Dumas; DW, Dave Wright; HP, Howard Penner; JB, Jeff Brock; JP, John Parker; KS, Ken Stephenson; RB, Robert Brooks; SL, Silvio Levy; YM, Yair Minsky.

Preface

To a topologist a teacup is the same as a bagel, but they are not the same to a geometer. By analogy, it is one thing to know the topology of a 3-manifold, another thing entirely to know its geometry — to find its shortest curves and their lengths, to make constructions with polyhedra, etc. In a word, we want to do geometry in the manifold just like we do geometry in euclidean space.

But do general 3-manifolds have "natural" metrics? For a start we might wonder when they carry one of the standards: the euclidean, spherical or hyperbolic metric. The latter is least known and not often taught; in the stream of mathematics it has always been something of an outlier. However it turns out that it is a big mistake to just ignore it! We now know that the interior of " most" compact 3-manifolds carry a hyperbolic metric.

It is the purpose of this book to explain the geometry of hyperbolic manifolds. We will examine both the existence theory and the structure theory.

Why embark on such a study? Well after all, we do live in three dimensions; our brains are specifically wired to see well in space. It seems perfectly reasonable if not compelling to respond to the challenge of understanding the range of possibilities. In particular, it is not at all established that our own universe is euclidean space, as many so like to believe.

I will briefly summarize the recent history of our subject. Although Poincaré recognized in 1881 that Möbius transformations extend from the complex plane to upper half-space, the development of the theory of three-dimensional hyperbolic manifolds had to wait for progress in three-dimensional topology. It was as late as the mid-1950s that Papakyriakopoulos confirmed the validity of Dehn's Lemma and the Loop Theorem. Once that occurred, the wraps were off.

In the early 1960s, while 3-manifold topology was booming ahead, the theory of kleinian groups was abruptly awoken from its long somnolence by a brilliant discovery of Lars Ahlfors. Kleinian groups are the discrete isometry groups of hyperbolic 3-space. Working (as always) in the context of complex analysis, Ahlfors discovered their finiteness property. This was followed by Mostow's contrasting discovery that closed hyperbolic manifolds of dimension $n \geq 3$ are uniquely determined up to isometry by their isomorphism class. This too came as a bombshell as it is false for $n = 2$. Then came Bers' study of quasifuchsian groups and his and Maskit's fundamental discoveries of "degenerate groups" as limits of them. Along a different line, Jørgensen developed the methods for dealing with sequences of kleinian groups, recognizing the existence of two distinct kinds of convergence which he called "algebraic" and "geometric". He also discovered a key class of examples, namely hyperbolic 3-manifolds that fiber over the circle.

It wasn't until the late 1960s that 3-manifold topology was sufficiently understood, most directly by Waldhausen's work, and the fateful marriage of 3-manifold topology to the complex analysis of the group action on $\mathbb{S}^2$ occurred. The first application was to the classification and analysis of geometrically finite groups and their quotient manifolds.

During the 1960s and 1970s, Riley discovered a slew of faithful representations of knot and link groups in $\mathrm{PSL}(2, \mathbb{C})$. Although these were seen as curiosities at the time, his examples pressed further the question of just what class of 3-manifolds did the hyperbolic manifolds represent? Maskit had proposed using his combination theorems to construct all hyperbolic manifolds from elementary ones. Yet Peter Scott pointed out that the combinations that were then feasible would construct only a limited class of 3-manifolds.

So by the mid-1970s there was a nice theory, part complex analysis, part three-dimensional geometry and topology, part algebra. Noone had the slightest idea as to what the scope of the theory really was. Did kleinian groups represent a large class of manifolds, or only a small sporadic class?

The stage (but not the players) was ready for the dramatic entrance in the mid-1970s of Thurston. He arrived with a proof that the interior of "most" compact 3-manifolds has a hyperbolic structure. He brought with him an amazingly original, exotic, and very powerful set of topological/geometrical tools for exploring hyperbolic manifolds. The subject of two- and three-dimensional topology and geometry was never to be the same again.

This book. Having witnessed at first hand the transition from a special topic in complex analysis to a subject of broad significance and application in mathematics, it seemed appropriate to write a book to record and explain the transformation. My idea was to try to make the subject accessible to beginning graduate students with minimal specific prerequisites. Yet I wanted to leave students with more than a routine compendium of elementary facts. Rather I thought students should see the big picture, as if climbing a watchtower to overlook the forest. Each student should end his or her studies having a personal response to the timeless question: What is this good for?

With such thoughts in mind, I have tried to give a solid introduction and at the same time to provide a broad overview of the subject as it is today. In fact today, the subject has reached a certain maturity. The characterization those compact manifolds whose interiors carry a hyperbolic structure is complete, the final step being provided by Perelman's recent confirmation of the Geometrization Conjecture. Attention turned to the analysis of structure of hyperbolic manifolds assuming only a finitely generated fundamental group. Within the past few years, the structure of these has been worked out as well. The three big conjectures left over from the 1960s and 1970s have been solved: tameness, density, and classification of the ends (ideal boundary components). If one is willing to climb the watchtower, the view is quite remarkable.

It is a challenge to carry out the plan as outlined. The foundation of the subject rests on elements of three-dimensional topology, hyperbolic geometry, and modern complex analysis. None of these are regularly covered in courses at most places.

I have attempted to meet the challenge as follows. The presentation of the basic facts is fairly rigorous. These are included in the first four chapters, plus the optional Chapters 7 and 8. These chapters include crash courses in three-manifold topology, covering surfaces and manifolds, quasiconformal mappings, and Riemann surface theory. With the basic information under our belts, Chapters 5 and 6 (as well as parts of Chapters 3 and 4) are expository, without most proofs. The reader will find there both the Hyperbolization Theorem and the newly discovered structural properties of general hyperbolic manifolds.

At the end of each chapter is a long section titled “Exercises and Explorations”. Some of these are genuine exercises and/or important additional information directly related to the material in the chapter. Others dig away a bit at the proofs of some of the theorems by introducing new tools they have required. Still others are included to point out various paths one can follow into the deeper forest and beauty spots one can find there. Thus there are not only capsule introductions to big fields like geometric group theory, but presentations of other more circumscribed topics that I (at least) find fascinating and relevant.

Acknowledgments. It is a great pleasure to thank the people who have helped bring the book to fruition.

First I want to acknowledge the essential contributions of my friend and colleague Troels Jørgensen. Over more than 25 years we walked in the forest together discussing and admiring the landscape our studies revealed. In particular we discussed the “universal properties” of Chapter 3 for years, until it was too late to publish them. Chapters 7 and 8 are based on his private lectures.

David Wright kindly computed a number of limit sets of kleinian groups, some never before seen, others adapted from pictures created for *Indra’s Pearls* [Mumford et al. 2002]. The extent of his contribution is evident from the list of figures. His pictures can be downloaded from www.okstate.edu/~wrightd/Marden together with computational details. In addition, David Dumas was willing to share his visualization of a Bers slice amidst the surrounding archipelago of discreteness components. It serves as the frontispiece. Jeff Brock contributed his pictures of algebraic and geometric limits that originally appeared in [Brock 2001b]; these too can be seen on `www.math.brown.edu/~brock`. The presence of the many artfully crafted pictures is a tangible expression of the mathematical beauty of the subject.

I am very grateful to Ken’ichi Ohshika for reading and commenting on an early draft and Dick Canary for reading several chapters of a later draft of the manuscript. Ian Agol, Ken Bromberg, Richard Evans, Sadayoshi Kojima, Howie Masur, Vlad Markovic, Yair Minsky, Peter Scott and Juan Souto as well as other mathematicians have been generous in responding to specific questions as well.

I could not have completed the book in the present form without the expert guidance and participation of Silvio Levy. He identified math problems, fixing some of them, properly handled the LaTeX formatting, improved the syntax, crafted the diagrams, and inserted the pictures.

I want to acknowledge the institutional support from the Forschungsinstitut für Mathematik at ETH in Zurich, the Maths Research Center, University of Warwick, and not least, from my own department, the School of Mathematics of the University of Minnesota. In my semester course Math 8380, I was able to present a solid introduction and overview of the subject based on the main points in the first six chapters.

I am grateful to Caroline Series for introducing me to the Press and for her enthusiasm for the project. Cambridge University Press in the person of David Tranah has shown great flexibility in keeping the retail price down and publishing standards high. Most importantly, David provided Silvio Levy as editor.

The nineteenth-century history. The history of noneuclidean geometry in the early nineteenth century is fascinating because of a host of conflicted issues concerning axiom systems in geometry, and the nature of physical space [Gray 1986; 2002].

Jeremy Gray [2002] writes:

> Few topics are as elusive in the history of mathematics as Gauss's claim to be a, or even the, discoverer of Non-Euclidean geometry. Answers to this conundrum often depend on unspoken, even shifting, ideas about what it could mean to make such a discovery. … [A]mbiguities in the theory of Fourier series can be productive in a way that a flawed presentation of a new geometry cannot be, because there is no instinctive set of judgments either way in the first case, but all manner of training, education, philosophy and belief stacked against the novelties in the second case.

Gray goes on to quote from Gauss's 1824 writings:

> …the assumption that the angle sum is less than 180° leads to a geometry quite different from Euclid's, logically coherent, and one that I am entirely satisfied with. It depends on a constant, which is not given a priori. The larger the constant, the closer the geometry to Euclid's.… The theorems are paradoxical but not self-contradictory or illogical.… All my efforts to find a contradiction have failed, the only thing that our understanding finds contradictory is that, if the geometry were to be true, there would be an absolute (if unknown to us) measure of length a priori. … As a joke I've even wished Euclidean geometry was not true, for then we would have an absolute measure of length a priori.

From his detailed study of the history, Gray's conclusion expressed in his recent Zurich lecture is that the birth of noneuclidean geometry should be attributed to the independently written foundational papers of Lobachevsky in 1829 and Bólyai in 1832. As expressed in [Milnor 1994, p. 246], those two were the first "with the courage to publish" accounts of the new theory. Still,

[f]or the first forty years or so of its history, the field of non-euclidean geometry existed in a kind of limbo, divorced from the rest of mathematics, and without any firm foundation.

This state of affairs changed upon Beltrami's introduction in 1868 of the methods of differential geometry, working with constant curvature surfaces in general. He gave the first global description of what we now call hyperbolic space. See [Gray 1986, p. 351], [Milnor 1994, p. 246], [Stillwell 1996, pp. 7–62].

It was Poincaré who brought two-dimensional hyperbolic geometry into the form we study today. He showed how it was relevant to topology, differential equations, and number theory. Again I quote Gray, in his translation of Poincaré's work of 1880 [Gray 1986, p. 268–9].

There is a direct connection between the preceding considerations and the non-Euclidean geometry of Lobachevskii. What indeed is a geometry? It is the study of a *group of operations* formed by the displacements one can apply to a figure without deforming it. In Euclidean geometry this group reduces to *rotations* and *translations*. In the pseudo-geometry of Lobachevskii it is more complicated... [Poincaré's emphasis].

As already mentioned, the first appearance of what we now call Poincaré's conformal model of noneuclidean space was in his seminal 1881 paper on kleinian groups. He showed that the action of Möbius transformations in the plane had a natural extension to a conformal action in the upper half-space model.

Actually the names "fuchsian" and "kleinian" for the isometry groups of two- and three-dimensional space were attached by Poincaré. However Poincaré's choice more reflects his generosity of spirit toward Fuchs and Klein than the mathematical reality. Klein himself objected to the name "fuchsian". His objection in turn prompted Poincaré to introduce the name "kleinian" for the discontinuous groups that do not preserve a circle. The more apt name would perhaps have been "Poincaré groups" to cover both cases. For the full story see [Gray 1986, §6.4].

So here we are today, nearly 125 years after Poincaré and approaching 200 after the initial ferment of ideas of Gauss, witnessing a full flowering of the vision and struggle for understanding of the nineteenth-century masters.

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