Matrix Analysis

Second Edition

Linear algebra and matrix theory are fundamental tools in mathematical and physical science, as well as fertile fields for research. This new edition of the acclaimed text presents results of both classic and recent matrix analysis using canonical forms as a unifying theme, and demonstrates their importance in a variety of applications.

The authors have thoroughly revised, updated, and expanded on the first edition. The book opens with an extended summary of useful concepts and facts and includes numerous new topics and features, such as:

- New sections on the singular value and CS decompositions
- New applications of the Jordan canonical form
- A new section on the Weyr canonical form
- Expanded treatments of inverse problems and of block matrices
- A central role for the von Neumann trace theorem
- A new appendix with a modern list of canonical forms for a pair of Hermitian matrices and for a symmetric–skew symmetric pair
- Expanded index with more than 3,500 entries for easy reference
- More than 1,100 problems and exercises, many with hints, to reinforce understanding and develop auxiliary themes such as finite-dimensional quantum systems, the compound and adjugate matrices, and the Loewner ellipsoid
- A new appendix provides a collection of problem-solving hints.

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To the matrix theory community

Contents

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Preface to the Second Edition

The basic structure of the first edition has been preserved in the second because it remains congruent with the goal of writing "a book that would be a useful modern treatment of a broad range of topics...[that] may be used as an undergraduate or graduate text and as a self-contained reference for a variety of audiences." The quotation is from the Preface to the First Edition, whose declaration of goals for the work remains unchanged.

What is different in the second edition?

The core role of canonical forms has been expanded as a unifying element in understanding similarity (complex, real, and simultaneous), unitary equivalence, unitary similarity, congruence, *congruence, unitary congruence, triangular equivalence, and other equivalence relations. More attention is paid to cases of equality in the many inequalities considered in the book. Block matrices are a ubiquitous feature of the exposition in the new edition.

Learning mathematics has never been a spectator sport, so the new edition continues to emphasize the value of exercises and problems for the active reader. Numerous 2-by-2 examples illustrate concepts throughout the book. Problem threads (some span several chapters) develop special topics as the foundation for them evolves in the text. For example, there are threads involving the adjugate matrix, the compound matrix, finite-dimensional quantum systems, the Loewner ellipsoid and the Loewner–John matrix, and normalizable matrices; see the index for page references for these threads. The first edition had about 690 problems; the second edition has more than 1,100. Many problems have hints; they may be found in an appendix that appears just before the index.

A comprehensive index is essential for a book that is intended for sustained use as a reference after initial use as a text. The index to the first edition had about 1,200 entries; the current index has more than 3,500 entries. An unfamiliar term encountered in the text should be looked up in the index, where a pointer to a definition (in Chapter 0 or elsewhere) is likely to be found.

New discoveries since 1985 have shaped the presentation of many topics and have stimulated inclusion of some new ones. A few examples of the latter are the Jordan

canonical form of a rank-one perturbation, motivated by enduring student interest in the *Google* matrix; a generalization of real normal matrices (normal matrices A such that $A\bar{A}$ is real); computable block matrix criteria for simultaneous unitary similarity or simultaneous unitary congruence; G. Belitskii's discovery that a matrix commutes with a Weyr canonical form if and only if it is block upper triangular and has a special structure; the discovery by K. C. O'Meara and C. Vinsonhaler that, unlike the corresponding situation for the Jordan canonical form, a commuting family can be simultaneously upper triangularized by similarity in such a way that any one specified matrix in the family is in Weyr canonical form; and canonical forms for congruence and *congruence.

Queries from many readers have motivated changes in the way that some topics are presented. For example, discussion of Lidskii's eigenvalue majorization inequalities was moved from a section primarily devoted to singular value inequalities to the section where majorization is discussed. Fortunately, a splendid new proof of Lidskii's inequalities by C. K. Li and R. Mathias became available and was perfectly aligned with Chapter 4's new approach to eigenvalue inequalities for Hermitian matrices. A second example is a new proof of Birkhoff's theorem, which has a very different flavor from the proof in the first edition.

Instructors accustomed to the order of topics in the first edition may be interested in a chapter-by-chapter summary of what is different in the new edition:

0. Chapter 0 has been expanded by about 75% to include a more comprehensive summary of useful concepts and facts. It is intended to serve as an as-needed reference. Definitions of terms and notations used throughout the book can be found here, but it has no exercises or problems. Formal courses and reading for self-study typically begin with Chapter 1.

1. Chapter 1 contains new examples related to similarity and the characteristic polynomial, as well as an enhanced emphasis on the role of left eigenvectors in matrix analysis.

2. Chapter 2 contains a detailed presentation of real orthogonal similarity, an exposition of McCoy's theorem on simultaneous triangularization, and a rigorous treatment of continuity of eigenvalues that makes essential use of both the *unitary* and *triangular* aspects of Schur's unitary triangularization theorem. Section 2.4 (Consequences of Schur's triangularization theorem) is almost twice the length of the corresponding section in the first edition. There are two new sections, one devoted to the singular value decomposition and one devoted to the CS decomposition. Early introduction of the singular value decomposition permits this essential tool of matrix analysis to be used throughout the rest of the book.

3. Chapter 3 approaches the Jordan canonical form via the Weyr characteristic; it contains an exposition of the Weyr canonical form and its unitary variant that were not in the first edition. Section 3.2 (Consequences of the Jordan canonical form) discusses many new applications; it contains 60% more material than the corresponding section in the first edition.

4. Chapter 4 now has a modern presentation of variational principles and eigenvalue inequalities for Hermitian matrices via subspace intersections. It contains an expanded treatment of inverse problems associated with interlacing and other

classical results. Its detailed treatment of unitary congruence includes Youla's theorem (a normal form for a square complex matrix A under unitary congruence that is associated with the eigenstructure of $A\bar{A}$), as well as canonical forms for conjugate normal, congruence normal, and squared normal matrices. It also has an exposition of recently discovered canonical forms for congruence and *congruence and new algorithms to construct a basis of a coneigenspace.

5. Chapter 5 contains an expanded discussion of norm duality, many new problems, and a treatment of semi-inner products that finds application in a discussion of finite-dimensional quantum systems in Chapter 7.

6. Chapter 6 has a new treatment of the "disjoint discs" aspect of Geršgorin's theorem and a reorganized discussion of eigenvalue perturbations, including differentiability of a simple eigenvalue.

7. Chapter 7 has been reorganized now that the singular value decomposition is introduced in Chapter 2. There is a new treatment of the polar decomposition, new factorizations related to the singular value decomposition, and special emphasis on row and column inclusion. The von Neumann trace theorem (proved via Birkhoff's theorem) is now the foundation on which many applications of the singular value decomposition are built. The Loewner partial order and block matrices are treated in detail with new techniques, as are the classical determinant inequalities for positive definite matrices.

8. Chapter 8 uses facts about left eigenvectors developed in Chapter 1 to streamline its exposition of the Perron–Frobenius theory of positive and nonnegative matrices.

D. Appendix D contains new explicit perturbation bounds for the zeroes of a polynomial and the eigenvalues of a matrix.

F. Appendix F tabulates a modern list of canonical forms for a pair of Hermitian matrices, or a pair of matrices, one of which is symmetric and the other is skew symmetric. These canonical pairs are applications of the canonical forms for congruence and *congruence presented in Chapter 4.

Readers who are curious about the technology of book making may be interested to know that this book began as a set of LATEX files created manually by a company in India from hard copy of the first edition. Those files were edited and revised using the Scientific WorkPlace® graphical user interface and typesetting system.

The cover art for the second edition was the result of a lucky encounter on a Delta flight from Salt Lake City to Los Angeles in spring 2003. The young man in the middle seat said he was an artist who paints abstract paintings that are sometimes mathematically inspired. In the course of friendly conversation, he revealed that his special area of mathematical enjoyment was linear algebra, and that he had studied *Matrix Analysis*. After mutual expressions of surprise at the chance nature of our meeting, and a pleasant discussion, we agreed that appropriate cover art would enhance the visual appeal of the second edition; he said he would send something to consider. In due course a packet arrived from Seattle. It contained a letter and a stunning 4.5- by 5-inch color photograph, identified on the back as an image of a 72- by 66-inch oil on canvas, painted in 2002. The letter said that "the painting is entitled *Surprised Again on the Diagonal* and is inspired by the recurring prevalence of the diagonal in math whether it be in geometry, analysis, algebra, set theory or logic. I think that it would

be an attractive addition to your wonderful book." Thank you, Lun-Yi Tsai, for your wonderful cover art!

A great many students, instructors, and professional colleagues have contributed to the evolution of this new edition since its predecessor appeared in 1985. Special thanks are hereby acknowledged to T. Ando, Wayne Barrett, Ignat Domanov, Jim Fill, Carlos Martins da Fonseca, Tatiana Gerasimova, Geoffrey Goodson, Robert Guralnick, Thomas Hawkins, Eugene Herman, Khakim Ikramov, Ilse Ipsen, Dennis C. Jespersen, Hideki Kosaki, Zhongshan Li, Teck C. Lim, Ross A. Lippert, Roy Mathias, Dennis Merino, Arnold Neumaier, Kevin O'Meara, Peter Rosenthal, Vladimir Sergeichuk, Wasin So, Hugo Woerdeman, and Fuzhen Zhang.

R.A.H.

Preface to the First Edition

Linear algebra and matrix theory have long been fundamental tools in mathematical disciplines as well as fertile fields for research in their own right. In this book, and in the companion volume, *Topics in Matrix Analysis*, we present classical and recent results of matrix analysis that have proved to be important to applied mathematics. The book may be used as an undergraduate or graduate text and as a self-contained reference for a variety of audiences. We assume background equivalent to a one-semester elementary linear algebra course and knowledge of rudimentary analytical concepts. We begin with the notions of eigenvalues and eigenvectors; no prior knowledge of these concepts is assumed.

Facts about matrices, beyond those found in an elementary linear algebra course, are necessary to understand virtually any area of mathematical science, whether it be differential equations; probability and statistics; optimization; or applications in theoretical and applied economics, the engineering disciplines, or operations research, to name only a few. But until recently, much of the necessary material has occurred sporadically (or not at all) in the undergraduate and graduate curricula. As interest in applied mathematics has grown and more courses have been devoted to advanced matrix theory, the need for a text offering a broad selection of topics has become more apparent, as has the need for a modern reference on the subject.

There are several well-loved classics in matrix theory, but they are not well suited for general classroom use, nor for systematic individual study. A lack of problems, applications, and motivation; an inadequate index; and a dated approach are among the difficulties confronting readers of some traditional references. More recent books tend to be either elementary texts or treatises devoted to special topics. Our goal was to write a book that would be a useful modern treatment of a broad range of topics.

One view of "matrix analysis" is that it consists of those *topics* in linear algebra that have arisen out of the needs of mathematical analysis, such as multivariable calculus, complex variables, differential equations, optimization, and approximation theory. Another view is that matrix analysis is an *approach* to real and complex linear

algebraic problems that does not hesitate to use notions from analysis – such as limits, continuity, and power series – when these seem more efficient or natural than a purely algebraic approach. Both views of matrix analysis are reflected in the choice and treatment of topics in this book. We prefer the term *matrix analysis* to *linear algebra* as an accurate reflection of the broad scope and methodology of the field.

For review and convenience in reference, Chapter 0 contains a summary of necessary facts from elementary linear algebra, as well as other useful, though not necessarily elementary, facts. Chapters 1, 2, and 3 contain mainly core material likely to be included in any second course in linear algebra or matrix theory: a basic treatment of eigenvalues, eigenvectors, and similarity; unitary similarity, Schur triangularization and its implications, and normal matrices; and canonical forms and factorizations, including the Jordan form, *LU* factorization, *QR* factorization, and companion matrices. Beyond this, each chapter is developed substantially independently and treats in some depth a major topic:

1. *Hermitian and complex symmetric matrices* (Chapter 4). We give special emphasis to variational methods for studying eigenvalues of Hermitian matrices and include an introduction to the notion of majorization.
2. *Norms on vectors and matrices* (Chapter 5) are essential for error analyses of numerical linear algebraic algorithms and for the study of matrix power series and iterative processes. We discuss the algebraic, geometric, and analytic properties of norms in some detail and make a careful distinction between those norm results for matrices that depend on the submultiplicativity axiom for matrix norms and those that do not.
3. *Eigenvalue location and perturbation results* (Chapter 6) for general (not necessarily Hermitian) matrices are important for many applications. We give a detailed treatment of the theory of Geršgorin regions, and some of its modern refinements, and of relevant graph theoretic concepts.
4. *Positive definite matrices* (Chapter 7) and their applications, including inequalities, are considered at some length. A discussion of the polar and singular value decompositions is included, along with applications to matrix approximation problems.
5. *Entry-wise nonnegative and positive matrices* (Chapter 8) arise in many applications in which nonnegative quantities necessarily occur (probability, economics, engineering, etc.), and their remarkable theory reflects the applications. Our development of the theory of nonnegative, positive, primitive, and irreducible matrices proceeds in elementary steps based on the use of norms.

In the companion volume, further topics of similar interest are treated: the field of values and generalizations; inertia, stable matrices, *M*-matrices and related special classes; matrix equations, Kronecker and Hadamard products; and various ways in which functions and matrices may be linked.

This book provides the basis for a variety of one- or two-semester courses through selection of chapters and sections appropriate to a particular audience. We recommend that an instructor make a careful preselection of sections and portions of sections of the book for the needs of a particular course. This would probably include Chapter 1, much of Chapters 2 and 3, and facts about Hermitian matrices and norms from Chapters 4 and 5.

Most chapters contain some relatively specialized or nontraditional material. For example, Chapter 2 includes not only Schur's basic theorem on unitary triangularization of a single matrix but also a discussion of simultaneous triangularization of families of matrices. In the section on unitary equivalence, our presentation of the usual facts is followed by a discussion of trace conditions for two matrices to be unitarily equivalent. A discussion of complex symmetric matrices in Chapter 4 provides a counterpoint to the development of the classical theory of Hermitian matrices. Basic aspects of a topic appear in the initial sections of each chapter, while more elaborate discussions occur at the ends of sections or in later sections. This strategy has the advantage of presenting topics in a sequence that enhances the book's utility as a reference. It also provides a rich variety of options to the instructor.

Many of the results discussed are valid or can be generalized to be valid for matrices over other fields or in some broader algebraic setting. However, we deliberately confine our domain to the real and complex fields where familiar methods of classical analysis as well as formal algebraic techniques may be employed.

Though we generally consider matrices to have complex entries, most examples are confined to real matrices, and no deep knowledge of complex analysis is required. Acquaintance with the arithmetic of complex numbers is necessary for an understanding of matrix analysis and is covered to the extent necessary in an appendix. Other brief appendices cover several peripheral, but essential, topics such as Weierstrass's theorem and convexity.

We have included many exercises and problems because we feel these are essential to the development of an understanding of the subject and its implications. The exercises occur throughout as part of the development of each section; they are generally elementary and of immediate use in understanding the concepts. We recommend that the reader work at least a broad selection of these. Problems are listed (in no particular order) at the end of each section; they cover a range of difficulties and types (from theoretical to computational) and they may extend the topic, develop special aspects, or suggest alternate proofs of major ideas. Significant hints are given for the more difficult problems. The results of some problems are referred to in other problems or in the text itself. We cannot overemphasize the importance of the reader's active involvement in carrying out the exercises and solving problems.

While the book itself is not about applications, we have, for motivational purposes, begun each chapter with a section outlining a few applications to introduce the topic of the chapter.

Readers who wish to consult alternative treatments of a topic for additional information are referred to the books listed in the References section following the appendices.

The list of book references is not exhaustive. As a practical concession to the limits of space in a general multitopic book, we have minimized the number of citations in the text. A small selection of references to papers – such as those we have explicitly used – does occur at the end of most sections accompanied by a brief discussion, but we have made no attempt to collect historical references to classical results. Extensive bibliographies are provided in the more specialized books we have referenced.

We appreciate the helpful suggestions of our colleagues and students who have taken the time to convey their reactions to the class notes and preliminary manuscripts

that were the precursors of the book. They include Wayne Barrett, Leroy Beasley, Bryan Cain, David Carlson, Dipa Choudhury, Risana Chowdhury, Yoo Pyo Hong, Dmitry Krass, Dale Olesky, Stephen Pierce, Leiba Rodman, and Pauline van den Driessche.

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