Chapter I

Fourier Series on $\mathbb{T}$

We denote by $\mathbb{R}$ the additive group of real numbers and by $\mathbb{Z}$ the subgroup consisting of the integers. The group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ where, as indicated by the notation, $2\pi\mathbb{Z}$ is the group of the integral multiples of 2π. There is an obvious identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$, which allows an implicit introduction of notions such as continuity, differentiability, etc., for functions on $\mathbb{T}$. The *Lebesgue measure* on $\mathbb{T}$ can be equally defined by means of the preceding identification: a function f is integrable on $\mathbb{T}$ if the corresponding 2π-periodic function, which we denote again by f, is integrable on $[0, 2\pi)$ and we set

$$\int_{\mathbb{T}} f(t)dt = \int_0^{2\pi} f(x)dx.$$

In other words, we consider the interval $[0, 2\pi)$ as a model for $\mathbb{T}$ and the Lebesgue measure dt on $\mathbb{T}$ is the restriction of the Lebesgue measure of $\mathbb{R}$ to $[0, 2\pi)$. The total mass of dt on $\mathbb{T}$ is equal to 2π and many of our formulas would be simpler if we normalized dt to have total mass 1, that is, if we replace it by $dx/2\pi$. Taking intervals on $\mathbb{R}$ as "models" for $\mathbb{T}$ is very convenient, however, and we choose to put $dt = dx$ in order to avoid confusion. We "pay" by having to write the factor $1/2\pi$ in front of every integral.

An all-important property of dt on $\mathbb{T}$ is its *translation invariance,* that is, for all $t_0 \in \mathbb{T}$ and f defined on $\mathbb{T}$,

$$\int f(t - t_0)dt = \int f(t)dt^{\dagger}.$$

[†]Throughout this chapter, integrals with unspecified limits of integration are taken over $\mathbb{T}$.

1 FOURIER COEFFICIENTS

1.1 We denote by $L^1(\mathbb{T})$ the space of all (equivalence† classes of) complex-valued, Lebesgue integrable functions on $\mathbb{T}$. For $f \in L^1(\mathbb{T})$ we put

$$\|f\|_{L^1} = \frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|dt.$$

It is well known that $L^1(\mathbb{T})$, with the norm so defined, is a Banach space.

DEFINITION: A *trigonometric polynomial* on $\mathbb{T}$ is an expression of the form

$$P \sim \sum_{n=-N}^{N} a_n e^{int}. \tag{1.1}$$

The numbers n appearing in (1.1) are called the frequencies of P; the largest integer n such that $|a_n| + |a_{-n}| \neq 0$ is called *the degree of* P. The values assumed by the index n are integers so that each of the summands in (1.1) is a function on $\mathbb{T}$. Since (1.1) is a finite sum, it represents a function, which we denote again by P, defined for each $t \in \mathbb{T}$ by

$$P(t) = \sum_{n=-N}^{N} a_n e^{int}. \tag{1.2}$$

Let P be defined by (1.2). Knowing the function P we can compute the coefficients a_n by the formula

$$a_n = \frac{1}{2\pi}\int P(t)e^{-int}dt \tag{1.3}$$

which follows immediately from the fact that for integers j,

$$\frac{1}{2\pi}\int e^{ijt}dt = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

Thus we see that the function P determines the expression (1.1) and there seems to be no point in keeping the distinction between the expression (1.1) and the function P; we shall consider trigonometric polynomials as both formal expressions and functions.

† $f \sim g$ if $f(t) = g(t)$ almost everywhere.

1.2 DEFINITION: A *trigonometric series* on $\mathbb{T}$ is an expression of the form

$$S \sim \sum_{n=-\infty}^{\infty} a_n e^{int}. \tag{1.4}$$

Again, n assumes integral values; however, the number of terms in (1.4) may be infinite and there is no assumption whatsoever about the size of the coefficients or about convergence. The conjugate† of the series (1.4) is, by definition, the series

$$\tilde{S} \sim \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) a_n e^{int},$$

where $\operatorname{sgn}(n) = 0$ if $n = 0$ and $\operatorname{sgn}(n) = n/|n|$ otherwise.

1.3 Let $f \in L^1(\mathbb{T})$. Motivated by (1.3) we define the nth Fourier coefficient of f by

$$\hat{f}(n) = \frac{1}{2\pi} \int f(t) e^{-int} dt. \tag{1.5}$$

DEFINITION: *The Fourier series* $S[f]$ *of a function* $f \in L^1(\mathbb{T})$ is the trigonometric series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}.$$

The series conjugate to $S[f]$ will be denoted by $\widetilde{S}[f]$ and referred to as the *conjugate Fourier series* of f. We shall say that a trigonometric series is a Fourier series if it is the Fourier series of some $f \in L^1(\mathbb{T})$.

1.4 We turn to some elementary properties of Fourier coefficients.

Theorem. *Let* $f, g \in L^1(\mathbb{T})$*, then*

(a) $\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$.

(b) For any complex number α

$$\widehat{(\alpha f)}(n) = \alpha \hat{f}(n).$$

† See Chapter III for motivation of the terminology.

(c) If $\bar{f}$ *is the complex conjugate*[‡] *of* f *then* $\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}$.

(d) Denote $f_\tau(t) = f(t-\tau)$, $\tau \in \mathbb{T}$*; then*

$$\hat{f}_\tau(n) = \hat{f}(n)e^{-in\tau}.$$

(e) $|\hat{f}(n)| \le \frac{1}{2\pi}\int |f(t)|dt = \|f\|_{L^1}$.

The proofs of (a) through (e) follow immediately from (1.5) and the details are left to the reader.

1.5 ***Corollary.*** *If* $f_j \in L^1(\mathbb{T})$, $j = 0, 1, \ldots,$ *and* $\|f_j - f_0\|_{L^1} \to 0$, *then* $\hat{f}_j(n) \to \hat{f}_0(n)$ *uniformly.*

1.6 ***Theorem.*** *Let* $f \in L^1(\mathbb{T})$, *assume* $\hat{f}(0) = 0$, *and define*

$$F(t) = \int_0^t f(\tau)d\tau.$$

Then F *is continuous,* 2π*-periodic, and*

$$\hat{F}(n) = \frac{1}{in}\hat{f}(n), \qquad n \neq 0. \tag{1.6}$$

PROOF: The continuity (and, in fact, the absolute continuity) of F is evident. The periodicity follows from

$$F(t+2\pi) - F(t) = \int_t^{t+2\pi} f(\tau)d\tau = 2\pi\hat{f}(0) = 0,$$

and (1.6) is obtained through integration by parts:

$$\hat{F}(n) = \frac{1}{2\pi}\int_0^{2\pi} F(t)e^{-int}dt = \frac{-1}{2\pi}\int_0^{2\pi} F'(t)\frac{1}{-in}e^{-int}dt = \frac{1}{in}\hat{f}(n).$$

◀

1.7 We now define the convolution operation in $L^1(\mathbb{T})$. The reader will notice the use of the group structure of $\mathbb{T}$ and of the invariance of dt in the subsequent proofs.

‡Defined by: $\bar{f}(t) = \overline{f(t)}$ for all $t \in \mathbb{T}$.

Theorem. *Let $f, g \in L^1(\mathbb{T})$. For almost all t, the function $f(t-\tau)g(\tau)$ is integrable (as a function of τ on $\mathbb{T}$) and, if we write*

$$h(t) = \frac{1}{2\pi}\int f(t-\tau)g(\tau)d\tau, \tag{1.7}$$

then $h \in L^1(\mathbb{T})$ and

$$\|h\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}. \tag{1.8}$$

Moreover

$$\hat{h}(n) = \hat{f}(n)\hat{g}(n) \quad \textit{for all } n. \tag{1.9}$$

PROOF: The functions $f(t-\tau)$ and $g(\tau)$, considered as functions of the two variables (t,τ), are clearly measurable, hence so is

$$F(t,\tau) = f(t-\tau)g(\tau).$$

For every τ, $F(t,\tau)$ is just a constant multiple of f_τ, hence integrable dt, and

$$\frac{1}{2\pi}\int\left(\frac{1}{2\pi}\int|F(t,\tau)|dt\right)d\tau = \frac{1}{2\pi}\int|g(\tau)|\,\|f\|_{L^1}d\tau = \|f\|_{L^1}\|g\|_{L^1}.$$

Hence, by the theorem of Fubini, $f(t-\tau)g(\tau)$ is integrable (over $(0,2\pi)$) as a function of τ for almost all t, and

$$\frac{1}{2\pi}\int|h(t)|dt = \frac{1}{2\pi}\int\left|\frac{1}{2\pi}\int F(t,\tau)d\tau\right|dt \le \frac{1}{4\pi^2}\iint|F(t,\tau)|dt\,d\tau = \|f\|_{L^1}\|g\|_{L^1},$$

which establishes (1.8). In order to prove (1.9) we write

$$\begin{aligned}\hat{h}(n) &= \frac{1}{2\pi}\int h(t)e^{-int}dt = \frac{1}{4\pi^2}\iint f(t-\tau)e^{-in(t-\tau)}g(\tau)e^{-in\tau}dt\,d\tau\\ &= \frac{1}{2\pi}\int f(t)e^{-int}dt\cdot\frac{1}{2\pi}\int g(\tau)e^{-in\tau}d\tau = \hat{f}(n)\hat{g}(n).\end{aligned}$$

As above, the change in the order of integration is justified by Fubini's theorem. ◀

1.8 DEFINITION: The *convolution* $f * g$ of the $(L^1(\mathbb{T}))$ functions f and g is the function h defined by (1.7). Using the star notation for the convolution, we can write (1.9) as

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n). \tag{1.10}$$

Theorem. *The convolution operation in $L^1(\mathbb{T})$ is commutative, associative, and distributive (with respect to the addition).*

PROOF: The change of variable $\vartheta = t - \tau$ gives

$$\frac{1}{2\pi}\int f(t-\tau)g(\tau)d\tau = \frac{1}{2\pi}\int g(t-\vartheta)f(\vartheta)d\vartheta,$$

that is,

$$f * g = g * f.$$

If $f_1, f_2, f_3 \in L^1(\mathbb{T})$, then

$$\begin{aligned}[(f_1 * f_2)*f_3](t) &= \frac{1}{4\pi^2}\iint f_1(t-u-\tau)f_2(u)f_3(\tau)du\,d\tau\\ &=\frac{1}{4\pi^2}\iint f_1(t-\omega)f_2(\omega-\tau)f_3(\tau)d\omega\,d\tau = [f_1 * (f_2 * f_3)](t).\end{aligned}$$

Finally, the distributive law

$$f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3$$

is evident from (1.7). ◀

1.9 ***Lemma.*** *Assume $f \in L^1(\mathbb{T})$ and let $\varphi(t) = e^{int}$ for some integer n. Then*

$$(\varphi * f)(t) = \hat{f}(n)e^{int}.$$

PROOF:

$$(\varphi * f)(t) = \frac{1}{2\pi}\int e^{in(t-\tau)}f(\tau)d\tau = e^{int}\frac{1}{2\pi}\int f(\tau)e^{-in\tau}d\tau.$$

◀

Corollary. *If $f \in L^1(\mathbb{T})$ and $k(t) = \sum_{-N}^{N} a_n e^{int}$, then*

$$(k * f)(t) = \sum_{-N}^{N} a_n\hat{f}(n)e^{int}. \tag{1.11}$$

EXERCISES FOR SECTION 1

1.1. Compute the Fourier coefficients of the following functions (defined by their values on $[-\pi, \pi)$):

(a)
$$f(t) = \begin{cases} \sqrt{2\pi} & |t| < \frac{1}{2} \\ 0 & \frac{1}{2} \le |t| \le \pi. \end{cases}$$

(b)
$$\Delta(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & 1 \le |t| \le \pi. \end{cases}$$

What relation do you see between f and Δ ?

(c)
$$g(t) = \begin{cases} 1 & -1 < t \le 0 \\ -1 & 0 < t < 1 \\ 0 & 1 \le |t|. \end{cases}$$

What relation do you see between g and Δ?

(d)
$$h(t) = t \qquad -\pi < t < \pi.$$

1.2. Remembering Euler's formulas

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}), \qquad \sin t = \frac{1}{2i}(e^{it} - e^{-it}),$$

or

$$e^{it} = \cos t + i \sin t\,,$$

show that the Fourier series of a function $f \in L^1(\mathbb{T})$ is formally equal to

$$\frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt)$$

where $A_n = \hat{f}(n) + \hat{f}(-n)$ and $B_n = i(\hat{f}(n) - \hat{f}(-n))$. Equivalently:

$$A_n = \frac{1}{\pi}\int f(t)\cos nt\,dt$$
$$B_n = \frac{1}{\pi}\int f(t)\sin nt\,dt.$$

Show also that if f is real valued, then A_n and B_n are all real; if f is even, that is, if $f(t) = f(-t)$, then $B_n = 0$ for all n; and if f is odd, that is, if $f(t) = -f(-t)$, then $A_n = 0$ for all n.

1.3. Show that if $S \sim \sum a_j \cos jt$, then $\tilde{S} \sim \sum a_j \sin jt$.

1.4. Let $f \in L^1(\mathbb{T})$ and let $P(t) = \sum_{-N}^{N} a_n e^{int}$. Compute the Fourier coefficients of the function fP.

1.5. Let $f \in L^1(\mathbb{T})$, let m be a positive integer, and write

$$f_{(m)}(t) = f(mt).$$

Show that

$$\widehat{f_{(m)}}(n) = \begin{cases} \hat{f}(\frac{n}{m}) & \text{if } m \mid n, \\ 0 & \text{if } m \not| \, n. \end{cases}$$

1.6. The trigonometric polynomial $\cos nt = \frac{1}{2}(e^{int} + e^{-int})$ is of degree n and has $2n$ zeros on $\mathbb{T}$. Show that no trigonometric polynomial of degree $n > 0$ can have more than $2n$ zeros on $\mathbb{T}$.
Hint: Identify $\sum_{-n}^{n} a_j e^{ijt}$ on $\mathbb{T}$ with $z^{-n} \sum_{-n}^{n} a_j z^{n+j}$ on $|z| = 1$.

1.7. Denote by C^* the multiplicative group of complex numbers different from zero. Denote by T^* the subgroup of all $z \in C^*$ such that $|z| = 1$. Prove that if G is a subgroup of C^* which is compact (as a set of complex numbers), then $G \subseteq T^*$.

1.8. Let G be a compact proper subgroup of $\mathbb{T}$. Prove that G is finite and determine its structure.
Hint: Show that G is discrete.

1.9. Let G be an infinite subgroup of $\mathbb{T}$. Prove that G is dense in $\mathbb{T}$.
Hint: The closure of G in $\mathbb{T}$ is a compact subgroup.

1.10. Let α be an irrational multiple of 2π. Prove that $\{n\alpha \pmod{2\pi}\}_{n\in\mathbb{Z}}$ is dense in $\mathbb{T}$.

1.11. Prove that a continuous homomorphism of $\mathbb{T}$ into C^* is necessarily given by an exponential function.
Hint: Use Exercise **1.7** to show that the map is into T^*; determine the mapping on "small" rational multiples of 2π and use Exercise **1.9**.

1.12. If E is a subset of $\mathbb{T}$ and $\tau_0 \in \mathbb{T}$, we define $E + \tau_0 = \{t + \tau_0 : t \in E\}$; and say that E is invariant under translation by τ if $E = E + \tau$. Show that, given a set E, the set of $\tau \in \mathbb{T}$ such that E is invariant under translation by τ is a subgroup of $\mathbb{T}$. Hence prove that if E is a measurable set on $\mathbb{T}$ and E is invariant under translation by infinitely many $\tau \in \mathbb{T}$, then either E or its complement has measure zero.
Hint: A set E of positive measure has points of density, that is, points τ such that $(2\varepsilon)^{-1}|E \cap (\tau - \varepsilon, \tau + \varepsilon)| \to 1$ as $\varepsilon \to 0$. ($|E_0|$ denotes the Lebesgue measure of E_0.)

1.13. If E and F are subsets of $\mathbb{T}$, we write

$$E + F = \{t + \tau : t \in E, \quad \tau \in F\}$$

and call $E+F$ the *algebraic sum* of E and F. We define similarly the sum of any finite number of sets. A set E is called *a basis for* $\mathbb{T}$ if there exists an integer N such that $E+E+\cdots+E$ (N times) is $\mathbb{T}$. Prove that every set E of positive measure on $\mathbb{T}$ is a basis.
Hint: Prove that if E contains an interval it is a basis. Using points of density prove that if E has positive measure then $E+E$ contains intervals.

1.14. Show that measurable proper subgroups of $\mathbb{T}$ have measure zero.

1.15. Show that measurable homomorphisms of $\mathbb{T}$ into C^* map it into T^*.

1.16. Let f be a measurable homomorphism of $\mathbb{T}$ into T^*. Show that for all values of n, except possibly one value, $\hat{f}(n)=0$.

2 SUMMABILITY IN NORM AND HOMOGENEOUS BANACH SPACES ON $\mathbb{T}$

2.1 We have defined the Fourier series of a function $f \in L^1(\mathbb{T})$ as a certain (formal) trigonometric series. The reader may wonder what is the point in the introduction of such formal series. After all, there is no more information in the (formal) expression $\sum_{-\infty}^{\infty} \hat{f}(n)e^{int}$ than there is in the simpler one $\{\hat{f}(n)\}_{-\infty}^{\infty}$ or the even simpler $\hat{f}$ (with the understanding that the function $\hat{f}$ is defined on the integers). As we shall see, both expressions, $\sum \hat{f}(n)e^{int}$ and $\hat{f}$, have their advantages; the main advantages of the series notation is that it indicates the way in which f can be reconstructed from $\hat{f}$. Much of this chapter and all of Chapter II will be devoted to clarifying the sense in which $\sum \hat{f}(n)e^{int}$ represents f. In this section we establish some of the main facts: we shall see that $\hat{f}$ determines f uniquely and we show how we can find f if we know $\hat{f}$.

Two very important properties of the Banach space $L^1(\mathbb{T})$ are the following:

H-1′ If $f \in L^1(\mathbb{T})$ and $\tau \in \mathbb{T}$, then

$$f_\tau(t) = f(t-\tau) \in L^1(\mathbb{T}) \quad \text{and} \quad \|f_\tau\|_{L^1} = \|f\|_{L^1}.$$

H-2′ The $L^1(\mathbb{T})$-valued function $\tau \mapsto f_\tau$ is continuous on $\mathbb{T}$, that is, for $f \in L^1(\mathbb{T})$ and $\tau_0 \in \mathbb{T}$

$$\lim_{\tau \to \tau_0} \|f_\tau - f_{\tau_0}\|_{L^1} = 0. \tag{2.1}$$

We shall refer to (H-1′) as the translation invariance of $L^1(\mathbb{T})$; it is an immediate consequence of the translation invariance of the measure

dt. In order to establish (H-2′) we notice first that (2.1) is clearly valid if f is a continuous function. Remembering that the continuous functions are dense in $L^1(\mathbb{T})$, we now consider an arbitrary $f \in L^1(\mathbb{T})$ and $\varepsilon > 0$. Let g be continuous on $\mathbb{T}$ such that $\|g - f\|_{L^1} < \varepsilon/2$; then

$$\begin{aligned}\|f_\tau - f_{\tau_0}\|_{L^1} &\le \|f_\tau - g_\tau\|_{L^1} + \|g_\tau - g_{\tau_0}\|_{L^1} + \|g_{\tau_0} - f_{\tau_0}\|_{L^1} \\ &= \|(f-g)_\tau\|_{L^1} + \|g_\tau - g_{\tau_0}\|_{L^1} + \|(g-f)_{\tau_0}\|_{L^1} \le \varepsilon + \|g_\tau - g_{\tau_0}\|_{L^1}.\end{aligned}$$

Hence $\overline{\lim}_{\tau\to\tau_0}\|f_\tau - f_{\tau_0}\|_{L^1} < \varepsilon$ and, ε being an arbitrary positive number, (H-2′) is established.

2.2 DEFINITION: A *summability kernel* is a sequence $\{k_n\}$ of continuous 2π-periodic functions satisfying:

(S-1) $$\frac{1}{2\pi}\int k_n(t)dt = 1.$$

(S-2) $$\frac{1}{2\pi}\int |k_n(t)|dt \le \text{const}.$$

(S-3) For all $0 < \delta < \pi$,

$$\lim_{n\to\infty}\int_\delta^{2\pi-\delta} |k_n(t)|dt = 0.$$

A *positive summability kernel* is one such that $k_n(t) \ge 0$ for all t and n. For positive kernels the assumption (S-2) is clearly redundant.

We consider also families k_r depending on a continuous parameter r instead of the discrete n. Thus the Poisson kernel $\mathbf{P}(r,t)$ (which we shall define at the end of this section) is defined for $0 \le r < 1$ and we replace in (S-3), as well as in the applications, the limit "$\lim_{n\to\infty}$" by "$\lim_{r\to 1}$".

The following lemma is stated in terms of vector-valued integrals. We refer to Appendix A for the definition and relevant properties.

Lemma. *Let B be a Banach space, φ a continuous B-valued function on $\mathbb{T}$, and $\{k_n\}$ a summability kernel. Then:*

$$\lim_{n\to\infty}\frac{1}{2\pi}\int k_n(\tau)\varphi(\tau)d\tau = \varphi(0).$$