EQUIVALENCE AND DUALITY FOR MODULE CATEGORIES

(with Tilting and Cotilting for Rings)

This book provides a unified approach to many of the theories of equivalence and duality between categories of modules that have transpired over the last 45 years. In particular, during the past dozen or so years many authors (including the authors of this book) have investigated relationships between categories of modules over a pair of rings that are induced by both covariant and contravariant representable functors, in particular by tilting and cotilting theories.

By here collecting and unifying the basic results of these investigations with innovative and easily understandable proofs, the authors' aim is to provide an aid to further research in this central topic in abstract algebra and a reference for all whose research lies in this field.

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EQUIVALENCE AND DUALITY FOR MODULE CATEGORIES

(with Tilting and Cotilting for Rings)

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Preface

Approximately forty-five years ago K. Morita presented the first major results
on equivalences and dualities between categories of modules over a pair of
rings. These results, which characterized an equivalence between the entire
categories of right (or left) modules over two rings as being represented by
the covariant Hom and tensor functors induced by a balanced bimodule that
is a so-called progenerator on either side, and which characterized a duality
between reasonably large subcategories of right and left modules over two
rings as being represented by the contravariant Hom functors induced by a
balanced bimodule that is an injective cogenerator on both sides, have come
to be known as the Morita theorems.

Morita's theorems on equivalence are exemplified by the equivalence of
the categories of right modules over a simple artinian ring and the right vector
spaces over its underlying division ring. More than a dozen years later the sec-
ond author, expanding on the relationship between the category of modules
generated by a simple module and the vector spaces over its endomorphism
ring, introduced the concept of a quasi-progenerator to characterize the equiv-
alences between a subcategory of right modules over one ring that is closed
under submodules, epimorphic images, and direct sums and the category of
all right modules over another ring. In the interim, employing the notion of
linear compactness, B. J. Müller had characterized the reflexive modules un-
der Morita duality and given a one-sided characterization of the bimodules
inducing these dualities. These results led to several investigations of the dual
notion of a quasi-progenerator that we refer to as a quasi-duality module.

In the early 1980s the notion of tilting modules and the tilting theorem for
finitely generated modules over artin algebras was introduced and polished in
papers by S. Brenner and M. C. R. Butler, D. Happel and C. M. Ringel, and
K. Bongartz to provide new insight and examples in the representation theory
of artin algebras. This has proved particularly effective in the study of algebras
of various representation types. The definition of a tilting module also applies
to modules over arbitrary rings and, in this setting, induces torsion theories in

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the categories of modules over two rings and a pair of equivalences between
the torsion and torsion-free parts of the torsion theories. These tilting modules
and equivalences are among the principal topics of this book.

Near the end of the 1980s C. Menini and A. Orsatti introduced a type of
module, which has come to be known as a ∗-module, that generalizes both
quasi-progenerators and tilting modules by inducing an equivalence between
a subcategory of modules over one ring that is closed under direct sums and
epimorphic images and a subcategory of modules over a second ring that is
closed under direct products and submodules.

At about the same time, the first author introduced a generalization of
Morita duality and the notions of cotilting modules and a cotilting theorem
for noetherian rings. This inspired several versions of cotilting modules and
cotilting theorems that are further principal topics of this book.

Just as progenerators, quasi-progenerators, and tilting modules are all
∗-modules, so are Morita duality modules, quasi-duality modules, and the
various flavors of cotilting modules, all dual versions of ∗-modules, a kind of
module that we call costar modules. As is so often the case in mathematics,
generalizations of a concept led to a better understanding of the concept. Here
we provide unified proofs regarding the various types of equivalence and du-
ality, and the modules that induce them, by approaching them via the general
notions of ∗-modules and costar modules. We feel that this approach should
yield improved accessibility to, and better understanding of, these concepts.
En route we present much of the relatively little that is known about how
properties of (the modules over) one of the rings in question are transformed
to the other ring under these various equivalences and dualities. We hope that
this exposition will inspire further research in this and related directions.

This book contains some of the work of many authors that has inspired
and, to a small extent been inspired by, our own research. We have made
an attempt to give credit where it is due, but surely we have inadvertently
omitted references to some works that should have been mentioned. To the
authors of these works, we offer our sincere apologies, and note that they
have undoubtedly been referred to in one or more of the papers and/or books
in our bibliography.

Acknowledgment

Colby wishes to express his gratitude for the hospitality of the University of
Iowa, where he is an independent scholar.