1

Some Module Theoretic Observations

We begin with a chapter consisting of several general facts involving various closure properties of certain categories of modules. These results are part of the background necessary for our future chapters, and we believe that they are of interest in themselves.

Throughout this book R denotes an associative ring with identity $1 \in R$, and Mod-R and R-Mod represent the categories of right and left R-modules and homomorphisms, while mod-R and R-mod denote their subcategories of finitely generated modules.

1.1. The Kernel of $\mathrm{Ext}^1_R(V, \_)$

For any R-module V we denote the kernel of $\mathrm{Ext}^1_R(V, \_)$ by $V^\perp$. Closure properties of $V^\perp$ are related to both homological and module-theoretic properties of V.

We denote the *projective dimension* of a module M by $\mathrm{proj.dim.}M$.

Proposition 1.1.1. *$V_R^\perp$ is closed under factors if and only if* $\mathrm{proj.dim.}V_R \le 1$.

Proof. If $V_R^\perp$ is closed under factors, $M \in$ Mod-R, and $E(M)$ is the injective envelope of M, then, since $E(M)/M \in V^\perp$, the exactness of the sequence

$$0 = \mathrm{Ext}^1_R(V, E(M)/M) \to \mathrm{Ext}^2_R(V, M) \to \mathrm{Ext}^2_R(V, E(M)) = 0$$

implies $\mathrm{proj.dim.}V_R \le 1$. Conversely, if $\mathrm{proj.dim.}V_R \le 1$ and $M \in V^\perp$ with K a submodule of M, we obtain $M/K \in V^\perp$ from the exactness of the sequence

$$0 = \mathrm{Ext}^1_R(V, M) \to \mathrm{Ext}^1_R(V, M/K) \to \mathrm{Ext}^2_R(V, K) = 0. \quad \blacksquare$$

Proposition 1.1.2. *If $V_R \in$ Mod-R is finitely presented, then $\mathrm{Ext}^1_R(V, \_)$ commutes with direct sums, so $V_R^\perp$ is closed under direct sums.*

Proof. If $\{M_\alpha\}_{\alpha\in A}$ is a family of modules in Mod-R, the natural monomorphism $\phi_V : \oplus_A \operatorname{Hom}_R(V, M_\alpha) \to \operatorname{Hom}_R(V, \oplus_A M_\alpha)$ is an isomorphism whenever V is finitely generated by [1, Exercise 16.3]. Moreover, ϕ induces natural homomorphisms $\theta_M : \oplus_A \operatorname{Ext}^1_R(M, M_\alpha) \to \operatorname{Ext}^1_R(M, \oplus_A M_\alpha)$. By hypothesis there is an exact sequence $0 \to K \to P \to V \to 0$ with P, K finitely generated and P projective. We obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc}\oplus_A\operatorname{Hom}_R(P, M_\alpha) & \to & \oplus_A\operatorname{Hom}_R(K, M_\alpha) & \to & \oplus_A\operatorname{Ext}^1_R(V, M_\alpha) & \to & 0\\ \downarrow \phi_P & & \downarrow \phi_K & & \downarrow \theta_V & & \\ \operatorname{Hom}_R(P, \oplus_A M_\alpha) & \to & \operatorname{Hom}_R(K, \oplus_A M_\alpha) & \to & \operatorname{Ext}^1_R(V, \oplus_A M_\alpha) & \to & 0\end{array}$$

from which the lemma follows. ∎

We note that a partial converse of this last result is found in the proof of Lemma 1.2 of [77].

Proposition 1.1.3. *If V_R is finitely generated and $V_R^\perp$ is closed under factors and direct sums, then V_R is finitely presented.*

Proof. We have $\operatorname{proj.dim.} V_R \le 1$ by Proposition 1.1.1; thus, since V_R is finitely generated, there is an exact sequence $0 \to L \to R^n \to V \to 0$, where L is projective. Hence, there is a split monomorphism $j : L \to R^{(X)}$ for some set X. By hypothesis $E(R)^{(X)} \in V^\perp$, so the composition of j with the inclusion i of $R^{(X)}$ into $E(R)^{(X)}$ has an extension to an element $f \in \operatorname{Hom}(R^n, E(R)^{(X)})$. Then $f(R^n) \subseteq E(R)^{(F)} \subseteq E(R)^{(X)}$ for some finite subset F of X. It follows that $j(L) \subseteq R^{(F)} \subseteq R^{(X)}$; therefore, since j is split monic, L is finitely generated. ∎

1.2. Gen(V) and Finiteness

We recall (see [1]) that for any collection $\mathcal{V}$ of R-modules, Gen($\mathcal{V}$) (gen($\mathcal{V}$)) denotes the full category of R-modules that are epimorphic images of (finite) direct sums of modules isomorphic to those in $\mathcal{V}$, and we let $\operatorname{Tr}_{\mathcal{V}}(M)$ denote the *trace of* $\mathcal{V}$ *in* M, the unique largest submodule of M that belongs to Gen($\mathcal{V}$). If $\mathcal{V}$ consists of a single module V_R we simply write Gen(V_R), and if $S = \operatorname{End}(V_R)$, then $\operatorname{Tr}_V(M)$ is the image of the canonical mapping $\nu_M : V \otimes_S \operatorname{Hom}_R(V, M) \to M$.

In order to characterize when Gen(V_R) is closed under direct products, we employ the following notions and lemmas.

Given that $\{M_\alpha\}_{\alpha\in A}$ is a family in Mod-R, for each $N \in R$-Mod we let

$$\eta_{\Pi_A M_\alpha, N} : (\Pi_A M_\alpha) \otimes_R N \longrightarrow \Pi_A(M_\alpha \otimes_R N)$$

denote the canonical mapping to obtain a natural transformation

$$\eta_{\Pi_A M_\alpha} : ((\Pi_A M_\alpha) \otimes_R \_) \longrightarrow \Pi_A(M_\alpha \otimes_R \_).$$

Lemma 1.2.1. *Suppose that $\{M_\alpha\}_{\alpha\in A}$ is a family in* Mod-R. *If ${}_RN$ is finitely generated (finitely presented), then the canonical homomorphism*

$$\eta_{\Pi_A M_\alpha, N} : (\Pi_A M_\alpha) \otimes_R N \longrightarrow \Pi_A(M_\alpha \otimes_R N)$$

is an epimorphism (isomorphism).

Proof. If $\{x_1, \ldots, x_n\}$ generate ${}_RN$, then any element of $M_\alpha \otimes_R N$ can be written in the form $\Sigma_i\, m_{\alpha i} \otimes x_i$.

Now assume that N is finitely presented and let $0 \to K \to P \to N \to 0$ be an exact sequence with P finitely generated and projective and K finitely generated. Then we have a commutative diagram

$$\begin{array}{ccccccc}
(\Pi_A M_\alpha) \otimes K & \to & (\Pi_A M_\alpha) \otimes P & \to & (\Pi_A M_\alpha) \otimes N & \to 0 \\
\eta_{\Pi_A M_\alpha, K} \downarrow & & \eta_{\Pi_A M_\alpha, P} \downarrow & & \eta_{\Pi_A M_\alpha, N} \downarrow & \\
\Pi_A(M_\alpha \otimes K) & \to & \Pi_A(M_\alpha \otimes P) & \to & \Pi_A(M_\alpha \otimes N) & \to 0
\end{array}$$

with exact rows, in which $\eta_{\Pi_A M_\alpha, K}$ is epic, and $\eta_{\Pi_A M_\alpha, P}$ is easily seen to be an isomorphism by naturalness of $\eta_{\Pi_A M_\alpha}$. Hence, by the Five Lemma, $\eta_{\Pi_A M_\alpha, N}$ is an isomorphism. ■

Identifying $R \otimes_R N = N$, we have the following result.

Lemma 1.2.2. *Let* $N \in R$-Mod. *Then the canonical homomorphism*

$$\eta_{R^A, N} : (R^A) \otimes_R N \longrightarrow N^A$$

is an epimorphism (isomorphism) for all sets A if and only if N is finitely generated (finitely presented).

Proof. The condition is sufficient in either case by Lemma 1.2.1. Conversely, letting $A = N$, if the diagonal element $(n)_{n\in N}$ is the image of some element $\Sigma_{i=1}^m (a_{ni})_{n\in N} \otimes_R x_i$, then, for all $n \in N$, $n = \Sigma_i a_{ni} x_i$. Thus, ${}_RN$ is finitely generated whenever $\eta_{R^N, N}$ is epic. Now supposing that $\eta_{R^A, N}$ is an isomorphism for all sets A, there is an exact sequence $0 \to K \to P \to N \to 0$

with P finitely generated and projective. Then both $\eta_{R^A,P}$ and $\eta_{R^A,N}$ are isomorphisms in the commutative diagram

$$\begin{array}{ccccccccc} & & R^A\otimes K & \longrightarrow & R^A\otimes P & \longrightarrow & R^A\otimes N & \to & 0 \\ & & \eta_{R^A,K}\downarrow & & \eta_{R^A,P}\downarrow & & \eta_{R^A,N}\downarrow & & \\ 0 & \to & K^A & \longrightarrow & P^A & \longrightarrow & N^A & \longrightarrow & 0 \end{array}$$

with exact rows. Hence, by the Snake Lemma, $\eta_{R^A,K}$ is an epimorphism, and so K is finitely generated. ■

Now we are in position to determine just when $\mathrm{Gen}(V_R)$ is closed under direct products.

Proposition 1.2.3. *The following statements about a module V_R with $S = \mathrm{End}(V_R)$ are equivalent:*

(a) $\mathrm{Gen}(V_R)$ contains V^A for all sets A;
(b) $\mathrm{Gen}(V_R)$ is closed under direct products;
(c) ${}_SV$ is finitely generated.

Proof. A module M_R is in $\mathrm{Gen}(V_R)$ if and only if the canonical trace mapping $\nu_M : \mathrm{Hom}_R(V, M)\otimes_S V \to M$ is epic. For any set A we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(V, V^A)\otimes_S V & \stackrel{\cong}{\to} & \mathrm{Hom}_R(V, V)^A\otimes_S V \\ \downarrow \nu_{V^A} & & \downarrow = \\ V^A & \stackrel{\eta_{S^A,V}}{\longleftarrow} & S^A\otimes_S V, \end{array}$$

so $(a) \Leftrightarrow (c)$ follows from Lemma 1.2.2.

$(b) \Rightarrow (a)$ is clear. For $(c) \Rightarrow (b)$, assume that ${}_SV$ is finitely generated and $\{M_\alpha\}_{\alpha\in A}$ belong to $\mathrm{Gen}(V_R)$. Then the composite of the canonical homomorphisms

$$\begin{aligned} \mathrm{Hom}_R(V, \Pi_A M_\alpha)\otimes_S V &\cong (\Pi_A \mathrm{Hom}_R(V, M_\alpha))\otimes_S V \\ &\stackrel{\eta}{\longrightarrow} \Pi_A(\mathrm{Hom}_R(V, M_\alpha)\otimes_S V) \\ &\stackrel{\Pi_A\nu_{M_\alpha}}{\longrightarrow} \Pi_A M_\alpha \end{aligned}$$

is epic by Lemma 1.2.1 and this composite is $\nu_{\Pi M_\alpha}$. ■

Next we obtain a mapping, in addition to the trace map, that determines whether a module belongs to $\mathrm{Gen}(V^A)$.

Lemma 1.2.4. *Let* $i : R_R \to V_R$ *be a homomorphism. If*

$$\mathrm{Hom}_R(i, M) : \mathrm{Hom}_R(V, M) \to \mathrm{Hom}_R(R, M)$$

is an epimorphism, then $M \in \mathrm{Gen}(V_R)$.

Proof. Let $\varphi : \mathrm{Hom}_R(R, M) \to M$ be the canonical isomorphism. Then, by hypothesis, for each $m \in M$ there is an $f_m \in \mathrm{Hom}_R(V, M)$ such that $m = \varphi(\mathrm{Hom}_R(i, M)(f_m)) = \varphi(f_m \circ i) = f_m(i(1))$, so $m \in \mathrm{Tr}_V(M)$. ∎

Let V_R be a fixed module with $S = \mathrm{End}(V_R)$. Let $\{x_\alpha\}_{\alpha \in A}$ be a generating set for ${}_SV$ and define $i : R_R \to V_R^A$ via $i(r) = (x_\alpha r)_{\alpha \in A}$. Plainly, $\mathrm{Ker}(i) = \mathrm{Ann}_R(V)$. Thus we have the exact sequence

$$0 \to \mathrm{Ann}_R(V) \to R_R \xrightarrow{i} V_R^A \to (V^A/i(R))_R \to 0.$$

For any $M \in \text{Mod–}R$, denote by i_M^* the composite of the homomorphisms

$$\mathrm{Hom}_R(V^A, M) \xrightarrow{\mathrm{Hom}(i,M)} \mathrm{Hom}_R(R, M) \xrightarrow{\cong} M.$$

Proposition 1.2.5. $M \in \mathrm{Gen}(V_R^A)$ *if and only if* i_M^* *is epic. In particular, if* V_R *is finitely generated over its endomorphism ring (and* A *is taken to be finite), then* i_M^* *is epic if and only if* $M \in \mathrm{Gen}(V_R)$.

Proof. Denote the class of $M \in \text{Mod-}R$ for which i_M^* is epic by $\mathcal{E}$. We first claim that $V_R \in \mathcal{E}$ and $\mathcal{E}$ is closed under epimorphic images. If $v \in V$, let $v = \Sigma_{j=1}^k s_{\alpha_j} x_{\alpha_j}$. Define $f \in \mathrm{Hom}_R(V^A, V)$ via $f((v_\alpha)) = \Sigma_{j=1}^k s_{\alpha_j} v_{\alpha_j}$. Then $i_V^*(f) = (f \circ i)(1) = v$. For the second assertion suppose that $M \xrightarrow{\eta} L$ is epic in R-Mod where i_M^* is also epic. Then, since $\eta \circ i_M^* = i_L^* \circ \mathrm{Hom}(V^A, \eta)$, i_L^* is also epic and our claim is proved.

Next we note that $\mathcal{E}$ is closed under arbitrary direct sums and products. Plainly, $\mathcal{E}$ is closed under arbitrary direct products and hence under finite direct sums. Let $M = \oplus_{\beta \in B} M_\beta$, where each $M_\beta \in \mathcal{E}$, and let $m \in M$. There is a finite subset B_0 of B such that if κ is the canonical inclusion $M_0 = \oplus_{\beta \in B_0} M_\beta \hookrightarrow M$, $m = \kappa(m')$, with $m' \in M_0$. But then there exists $f \in \mathrm{Hom}_R(V^A, M_0)$ such that $m = \kappa(m') = \kappa(i_{M_0}^*(f)) = i_M^*(\mathrm{Hom}(V^A, \kappa)(f))$.

Now, $\mathrm{Gen}(V^A) \subseteq \mathcal{E}$ follows from what we have proved thus far. The reverse inclusion follows immediately from Lemma 1.2.4. ∎

Since the class of modules M for which i_M^* is epic is clearly closed under direct products, Proposition 1.2.5 implies that $\mathrm{Gen}(V^A)$ is closed under direct products; hence, we have the following corollary by Proposition 1.2.3.

Corollary 1.2.6. *For any V_R, if V is generated over its endomorphism ring by $|A|$ elements, then V^A is finitely generated over its endomorphism ring.*

A module V_R is *small* if, as is the case for a finitely generated module, $\mathrm{Hom}_R(V, \oplus_A M_\alpha) \cong \oplus_A \mathrm{Hom}_R(V, M_\alpha)$, canonically, for all $\{M_\alpha\}_A$ in Mod-R. A module V_R is *self-small* if $\mathrm{Hom}_R(V, V^{(A)}) \cong \mathrm{Hom}_R(V, V)^{(A)}$, canonically, for all sets A. This notion is a key element in the proof of the following proposition due to J. Trlifaj [78].

Proposition 1.2.7. *If* $\mathrm{Hom}_R(V, \_)$ *commutes with direct limits (with directed index sets) of modules in* $\mathrm{Gen}(V_R)$, *then* V_R *is finitely generated.*

Proof. First we note that, since $V^{(A)} = \varinjlim V^{(F)}$ such that F is a finite subset of A [69, pp. 44–45], we have, by hypothesis, $\mathrm{Hom}_R(V, V^{(A)}) = \mathrm{Hom}_R(V, \varinjlim V^{(F)}) \cong \varinjlim \mathrm{Hom}_R(V, V^{(F)}) \cong \mathrm{Hom}_R(V, V)^{(A)}$; thus, V is self-small. Let $V = \sum_A x_\alpha R$ and let $\iota_\alpha : V \to V^{(A)}$ ($\alpha \in A$) be the canonical injections. Then, identifying $x_\alpha = \iota_a x_\alpha$, since $V \cong (\oplus_A \iota_\alpha x_\alpha R)/K$, we have a monomorphism

$$\varphi : V \to V^{(A)}/K$$

with $K \leq \oplus_A x_\alpha R$ and

$$\varphi : x_\alpha \mapsto x_\alpha + K.$$

Then, letting $\{K_i\}_{i\in I}$ denotes the finitely generated submodules of K, with canonical epimorphisms $\gamma_i : V^{(A)}/K_i \to V^{(A)}/K$ ($i \in I$),

$$(V^{(A)}/K, \{\gamma_i\}_I) = \varinjlim V^{(A)}/K_i.$$

Now, by hypothesis,

$$(\mathrm{Hom}_R(V, V^{(A)}/K), \{\mathrm{Hom}_R(V, \gamma_i)\}_I) = \varinjlim \mathrm{Hom}_R(V, V^{(A)}/K_i)$$

so that (see [69, Theorem 2.17])

$$\mathrm{Hom}_R(V, V^{(A)}/K) = \cup_I \operatorname{Im} \mathrm{Hom}_R(V, \gamma_i).$$

Thus there is an $i \in I$ and a $\varphi_i \in \mathrm{Hom}_R(V, V^{(A)}/K_i)$ with

$$\varphi = \gamma_i \varphi_i.$$

There is a finite set $F \subseteq A$ such that $K_i \subseteq V^{(F)}$, and hence

$$V^{(A)}/K_i = V^{(F)}/K_i \oplus V^{(A\setminus F)}.$$

So since V is self-small

$$\operatorname{Im}\varphi_i \subseteq V^{(H)}/K_i$$

for some finite set H with $F \subseteq H \subseteq A$. Now we have

$$\varphi(V) \subseteq \gamma_i(V^{(H)}/K_i) = (V^{(H)} + K)/K,$$

so, for each $\alpha \in A$, there is a $v_\alpha \in V^{(H)}$ such that

$$v_\alpha + K = \varphi(x_\alpha) = x_\alpha + K.$$

But then

$$v_\alpha \in V^{(H)} \cap (\oplus_A x_\alpha R) = \oplus_H x_\alpha R,$$

and we have

$$\operatorname{Im}\varphi \subseteq (\oplus_H x_\alpha R + K)/K \subseteq \operatorname{Im}\varphi;$$

thus, $V \cong \operatorname{Im}\varphi$ is finitely generated. ■

Another closure property of Gen(V_R) forces V to be flat over its endomorphism ring.

Proposition 1.2.8. *Suppose* $V \in$ Mod-R *and* $S = \operatorname{End}(V_R)$. *If* Gen($V_R$) *is closed under submodules, then* ${}_SV$ *is flat.*

Proof. Recall [1, Lemma 19.19] that ${}_SV$ is flat if and only if for every relation

$$\sum_{i=1}^{m} s_i x_i = 0 \qquad (s_i \in S, \quad x_i \in V)$$

there exist $y_j \in V, \sigma_{ij} \in S, \quad 1 \le i \le m, 1 \le j \le n,$ such that for all $1 \le i \le m$ and $1 \le j \le n$

$$\sum_{j=1}^{n} \sigma_{ij} y_j = x_i \text{ and } \sum_{i=1}^{m} s_i \sigma_{ij} = 0.$$

So suppose we do have

$$\sum_{i=1}^{m} s_i x_i = 0 \qquad (s_i \in S, \quad x_i \in V);$$

let $\pi_j : V^m \to V, \quad 1 \le j \le m$, be the canonical projections and let

$$K = \operatorname{Ker} d$$

where d is the homomorphism $d : V^{(m)} \to V$ defined by

$$d : z \mapsto \sum_{i=1}^{m} s_i \pi_i(z), \quad z \in V^{(m)}.$$

Then $x = (x_1, \dots, x_m) \in K$ and so, since V generates K, there exist

$$f_j : V \to K, \quad \text{and} \quad y_j \in V, \quad 1 \le j \le n$$

such that

$$x = \sum_{j=1}^{n} f_j y_j.$$

Now let

$$\sigma_{ij} = \pi_i f_j \in S, \qquad 1 \le i \le m,\ 1 \le j \le n,$$

to obtain

$$\sum_{j=1}^{n} \sigma_{ij} y_j = \pi_j(x) = x_i, \qquad 1 \le i \le m,$$

and for each $u \in V$

$$\sum_{i=1}^{m} s_i \sigma_{ij} u = \sum_{i=1}^{m} s_i \pi_i f_j u = d(f_j u) = 0, \qquad 1 \le j \le n. \quad \blacksquare$$

1.3. Add(V_R) and Prod(V_R)

We denote the subcategories of Mod-R consisting of all direct summands of a direct sum, respectively, a direct product, of copies of a module V_R by *Add*(V_R), respectively, by *Prod*(V_R).

According to [1, Theorems 19.20 and 28.4], if S is a left coherent right perfect ring, then every direct product of projective right S-modules is projective, that is, belongs to Add(S_S). (This result and its converse are due to S. Chase [12], who also proved that if every direct product of copies of S_S is projective, then S is a left coherent right perfect ring.) On the other hand we have

Lemma 1.3.1. *If S is a left coherent right perfect ring, then every projective right S-module belongs to* Prod(S_S).

Proof. Letting $J = J(S)$, suppose that P_S is projective and $P/PJ = \oplus_{\alpha \in A} T_\alpha$ with each T_α simple. Let $Q = S_S^A$. Then $QJ \le J^A$ and $\oplus_{\alpha \in A} T_\alpha$ is isomorphic to a direct summand of $(S/J)^A \cong S^A/J^A$, and so there is an epimorphism $Q \to P/PJ$. Thus by [1, Lemma 17.17] P, the projective cover of P/PJ, is isomorphic to a direct summand of Q. ■

This last lemma and the paragraph preceding it tell us that, if S is a left coherent right perfect ring, then $\mathrm{Add}(S_S) = \mathrm{Prod}(S_S)$.

Proposition 1.3.2. *Let V_R be a self-small module with* $\mathrm{End}(V_R) = S$. *If S is left coherent and right perfect, and ${}_SV$ is finitely presented, then* $\mathrm{Prod}(V_R) = \mathrm{Add}(V_R)$.

Proof. Since V is self-small

$$\mathrm{Hom}_R(V, \_) : \mathrm{Add}(V_R) \rightleftarrows \mathrm{Add}(S_S) : (\_ \otimes_S V)$$

is an equivalence of categories. But $\mathrm{Hom}_R(V, \_)$ commutes with direct products and, by Lemma 1.2.1, so does $(\_ \otimes_S V)$. Thus the proposition follows from the fact that $\mathrm{Add}(S_S) = \mathrm{Prod}(S_S)$. ■

A ring R is an *artin algebra* if its center K is an artinian ring and R is finitely generated as a K-module. Any finitely generated module over an artin algebra is finitely generated over its endomorphism ring, which is also an artin algebra. Thus we have

Corollary 1.3.3. *If V_R is a finitely generated module over an artin algebra R, then* $\mathrm{Prod}(V_R) = \mathrm{Add}(V_R)$.

Note that we have only used one implication of Chase's theorem. Using his full theorem, H. Krause and M. Saorín showed in [53] that a self-small module V_R with $S = \mathrm{End}(V_R)$ has $\mathrm{Add}(V_R)$ closed under direct products if and only if S is a left coherent right perfect ring and ${}_SV$ is finitely presented. In view of Proposition 1.3.2 this is equivalent to $\mathrm{Prod}(V_R) = \mathrm{Add}(V_R)$.

1.4. Torsion Theory

Definition 1.4.1. If $\mathcal{C}$ is an abelian category, a *torsion theory* in $\mathcal{C}$ is a pair of classes of objects $(\mathcal{T}, \mathcal{F})$ of $\mathcal{C}$ such that

(1) $\mathcal{T} = \{T \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(T, F) = 0 \text{ for all } F \in \mathcal{F}\}$,
(2) $\mathcal{F} = \{F \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(T, F) = 0 \text{ for all } T \in \mathcal{T}\}$,

(3) for each $X \in \mathcal{C}$ there is a subobject T of X such that

$$T \in \mathcal{T} \quad \text{and} \quad X/T \in \mathcal{F}.$$

When this is the case, the objects in $\mathcal{T}$ are called *torsion* objects, the elements of $\mathcal{F}$ are called *torsion-free* objects, and if the object T of (3) is unique, we denote it by $\tau(X)$ and call it the *torsion subobject* of X.

Suppose that $\mathcal{C}$ is a full subcategory of Mod-R that is closed under submodules, epimorphic images, extensions, direct sums, and direct products. If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, then it follows that $\mathcal{T}$ is closed under epimorphic images and direct sums, $\mathcal{F}$ is closed under submodules and direct products, and both are closed under extensions. A class $\mathcal{T}$ $(\mathcal{F})$ of modules in $\mathcal{C}$ with these closure properties is called a *torsion (torsion-free) class* in $\mathcal{C}$. Then one easily verifies

Proposition 1.4.2. *Let $\mathcal{C}$ be a full subcategory of* Mod-R *that is closed under submodules, epimorphic images, extensions, direct sums, and direct products.*

(1) If $\mathcal{T}$ is a torsion class in $\mathcal{C}$, then $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, where $\mathcal{F} = \{F \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(T, F) = 0$ for all $T \in \mathcal{T}\}$.
(2) If $\mathcal{F}$ is a torsion-free class in $\mathcal{C}$, then $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, where $\mathcal{T} = \{T \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(T, F) = 0$ for all $F \in \mathcal{F}$.

Dual to Gen($\mathcal{V}$), if $\mathcal{V}$ is a class of R-modules, Cogen *(V)* (cogen($\mathcal{V}$)) consists of the R-modules that embed in (finite) direct products of modules isomorphic to members of $\mathcal{V}$, and the *reject of* $\mathcal{V}$ *in* M is $\mathrm{Rej}_{\mathcal{V}}(M)$, the intersection of the kernels of all maps from M into members of $\mathcal{V}$.

Proposition 1.4.3. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in* Mod-R *and M a module in* Mod-R. *Then*

$$\mathrm{Tr}_{\mathcal{T}}(M) = \mathrm{Rej}_{\mathcal{F}}(M).$$

Proof. That $\mathrm{Tr}_{\mathcal{T}}(M) \subseteq \mathrm{Rej}_{\mathcal{F}}(M)$ follows from $\mathrm{Hom}_R(T, F) = 0$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$. But since $\mathrm{Tr}_{\mathcal{T}}(M) \in \mathcal{T}$ and $\mathcal{T}$ is closed under extensions, $M/\mathrm{Tr}_{\mathcal{T}}(M) \in \mathcal{F}$ and hence $\mathrm{Rej}_{\mathcal{F}}(M) \subseteq \mathrm{Tr}_{\mathcal{T}}(M)$. ■

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in Mod-R, we let

$$\tau_{\mathcal{T}}(M) = \mathrm{Tr}_{\mathcal{T}}(M) = \mathrm{Rej}_{\mathcal{F}}(M)$$