Measure Theory and Filtering

Introduction and Applications

The estimation of noisily observed states from a sequence of data has traditionally incorporated ideas from Hilbert spaces and calculus-based probability theory. As conditional expectation is the key concept, the correct setting for filtering theory is that of a probability space. Graduate engineers, mathematicians, and those working in quantitative finance wishing to use filtering techniques will find in the first half of this book an accessible introduction to measure theory, stochastic calculus, and stochastic processes, with particular emphasis on martingales and Brownian motion. Exercises are included, solutions to which are available from www.cambridge.org. The book then provides an excellent user's guide to filtering: basic theory is followed by a thorough treatment of Kalman filtering, including recent results that exend the Kalman filter to provide parameter estimates. These ideas are then applied to problems arising in finance, genetics, and population modelling in three separate chapters, making this a comprehensive resource for both practitioners and researchers.

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Preface

Traditional courses for engineers in filtering and signal processing have been based on elementary linear algebra, Hilbert space theory and calculus. However, the key objective underlying such procedures is the (recursive) estimation of indirectly observed states given observed data. This means that one is discussing conditional expected values, given the observations. The correct setting for conditional expected value is in the context of measurable spaces equipped with a probability measure, and the initial object of this book is to provide an overview of required measure theory. Secondly, conditional expectation, as an inverse operation, is best formulated as a form of Bayes' Theorem. A mathematically pleasing presentation of Bayes' theorem is to consider processes as being initially defined under a "reference probability." This is an idealized probability under which all the observations are independent and identically distributed. The reference probability is a much nicer measure under which to work. A suitably defined change of measure then transforms the distribution of the observations to their real world form. This setting for the derivation of the estimation and filtering results enables more general results to be obtained in a transparent way.

The book commences with a leisurely and intuitive introduction to σ-fields and the results in measure theory that will be required.

The first chapter also discusses random variables, integration and conditional expectation.

Chapter 2 introduces stochastic processes, with particular emphasis on martingales and Brownian motion.

Stochastic calculus is developed in Chapter 3 and techniques related to changing probability measures are described in Chapter 4.

The change of measure method is the basic technique used in this book.

The second part of the book commences with a treatment of Kalman filtering in Chapter 5. Recent results, which extend the Kalman filter and enable parameter estimates to be obtained, are included. These results are applied to financial models in Chapter 6. The final two chapters give some filtering applications to genetics and population models.

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