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0521837758 - The Cauchy-Schwarz Master Class: An Introduction to the Art of  
Mathematical Inequalities

J. Michael Steele

Frontmatter

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## THE CAUCHY–SCHWARZ MASTER CLASS

This lively, problem-oriented text is designed to coach readers toward mastery of the most fundamental mathematical inequalities. With the Cauchy–Schwarz inequality as the initial guide, the reader is led through a sequence of fascinating problems whose solutions are presented as they might have been discovered — either by one of history’s famous mathematicians or by the reader. The problems emphasize beauty and surprise, but along the way readers will find systematic coverage of the geometry of squares, convexity, the ladder of power means, majorization, Schur convexity, exponential sums, and the inequalities of Hölder, Hilbert, and Hardy.

The text is accessible to anyone who knows calculus and who cares about solving problems. It is well suited to self-study, directed study, or as a supplement to courses in analysis, probability, and combinatorics.

J. Michael Steele is C. F. Koo Professor of Statistics at the Wharton School, University of Pennsylvania. He is the author of more than 100 mathematical publications, including the books *Probability Theory and Combinatorial Optimization* and *Stochastic Calculus and Financial Applications*. He is also the founding editor of the *Annals of Applied Probability*.

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J. MICHAEL STEELE

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PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE  
The Pitt Building, Trumpington Street, Cambridge, United Kingdom  
And the Mathematical Association of America

CAMBRIDGE UNIVERSITY PRESS  
The Edinburgh Building, Cambridge CB2 2RU, UK  
40 West 20th Street, New York, NY 10011-4211, USA  
477 Williamstown Road, Port Melbourne, VIC 3207, Australia  
Ruiz de Alarcón 13, 28014 Madrid, Spain  
Dock House, The Waterfront, Cape Town 8001, South Africa  
<http://www.cambridge.org>

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First published 2004

Printed in the United States of America

*Typeface* Computer Modern 10/13 pt.     *System* L<sup>A</sup>T<sub>E</sub>X 2<sub>ε</sub> [AU]

*A catalog record for this book is available from the British Library.*

*Library of Congress Cataloging in Publication data available*

ISBN 0 521 83775 8 hardback  
ISBN 0 521 54677 X paperback

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## Preface

In the fine arts, a master class is a small class where students and coaches work together to support a high level of technical and creative excellence. This book tries to capture the spirit of a master class while providing coaching for readers who want to refine their skills as solvers of problems, especially those problems dealing with mathematical inequalities.

The most important prerequisite for benefiting from this book is the desire to master the craft of discovery and proof. The formal requirements are quite modest. Anyone with a solid course in calculus is well prepared for almost everything to be found here, and perhaps half of the material does not even require calculus. Nevertheless, the book develops many results which are rarely seen, and even experienced readers are likely to find material that is challenging and informative.

With the Cauchy–Schwarz inequality as the initial guide, the reader is led through a sequence of interrelated problems whose solutions are presented as they might have been discovered — either by one of history’s famous mathematicians or by the reader. The problems emphasize beauty and surprise, but along the way one finds systematic coverage of the geometry of squares, convexity, the ladder of power means, majorization, Schur convexity, exponential sums, and all of the so-called classical inequalities, including those of Hölder, Hilbert, and Hardy.

To solve a problem is a very human undertaking, and more than a little mystery remains about how we best guide ourselves to the discovery of original solutions. Still, as George Pólya and others have taught us, there are principles of problem solving. With practice and good coaching we can all improve our skills. Just like singers, actors, or pianists, we have a path toward a deeper mastery of our craft.

## ACKNOWLEDGMENTS

The initial enthusiasm of Herb Wilf and Theodore Körner propelled this project into being, and they deserve my special thanks. Many others have also contributed in essential ways over a period of years. In particular, Cynthia Cronin-Kardon provided irreplaceable library assistance, and Steve Skillen carefully translated almost all of the figures into PSTRICKS. Don Albers, Lauren Cowles, and Patrick Kelly all provided wise editorial advice which was unfailingly accepted. Patricia Steele ceded vast stretches of our home to ungainly stacks of paper and helped in many other ways.

For their responses to my enquiries and their comments on special parts of the text, I am pleased to thank Tony Cai, Persi Diaconis, Dick Dudley, J.-P. Kahane, Kirin Kedlaya, Hojoo Lee, Lech Maliganda, Zhihua Qian, Bruce Reznick, Paul Shaman, Igor Sharplinski, Larry Shepp, Huili Tang, and Rick Vitale. Many others kindly provided preprints, reprints, or pointers to their work or the work of others.

For their extensive comments covering the whole text (and in some cases in more than one version), I owe great debts to Cengiz Belentepe, Claude Dellacherie, Jirka Matoušek, Xioli Meng, and Nicholas Ward.