Discrete group actions

The genesis of analytic number theory formally began with the epoch making memoir of Riemann (1859) where he introduced the zeta function,
\[ \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\Re(s) > 1), \]
and obtained its meromorphic continuation and functional equation
\[ \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s), \quad \Gamma(s) = \int_{0}^{\infty} e^{-u} u^{s-1} du. \]
Riemann showed that the Euler product representation
\[ \zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \]
together with precise knowledge of the analytic behavior of \( \zeta(s) \) could be used to obtain deep information on the distribution of prime numbers.

One of Riemann’s original proofs of the functional equation is based on the Poisson summation formula
\[ \sum_{n \in \mathbb{Z}} f(ny) = y^{-1} \sum_{n \in \mathbb{Z}} \hat{f}(ny^{-1}), \]
where \( f \) is a function with rapid decay as \( y \to \infty \) and
\[ \hat{f}(y) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i ty} dt, \]
is the Fourier transform of \( f \). This is proved by expanding the periodic function
\[ F(x) = \sum_{n \in \mathbb{Z}} f(x + n). \]
in a Fourier series. If $f$ is an even function, the Poisson summation formula may be rewritten as

$$\sum_{n=1}^{\infty} f(ny^{-1}) = y \sum_{n=1}^{\infty} \hat{f}(ny) - \frac{1}{2} (y \hat{f}(0) - f(0)), $$

from which it follows that for $\Re(s) > 1$,

$$\zeta(s) \int_0^\infty f(y)y^{s-1} \, dy = \int_0^\infty \sum_{n=1}^{\infty} f(ny)y^{s-1} \, dy$$

$$= \int_1^\infty \sum_{n=1}^{\infty} (f(ny)y^s + f(ny^{-1})y^{-s}) \, dy$$

$$= \int_1^\infty \sum_{n=1}^{\infty} (f(ny)y^s + \hat{f}(ny)y^{1-s}) \, dy - \frac{1}{2} \left( \frac{f(0)}{s} + \frac{\hat{f}(0)}{1-s} \right).$$

If $f(y)$ and $\hat{f}(y)$ have sufficient decay as $y \to \infty$, then the integral above converges absolutely for all complex $s$ and, therefore, defines an entire function of $s$. Let

$$\tilde{f}(s) = \int_0^\infty f(y)y^s \, dy$$

denote the Mellin transform of $f$, then we see from the above integral representation and the fact that $\hat{f}(y) = f(-y) = f(y)$ (for an even function $f$) that

$$\zeta(s) \tilde{f}(s) = \zeta(1-s) \tilde{f}(1-s).$$

Choosing $f(y) = e^{-\pi y^2}$, a function with the property that it is invariant under Fourier transform, we obtain Riemann’s original form of the functional equation. This idea of introducing an arbitrary test function $f$ in the proof of the functional equation first appeared in Tate’s thesis (Tate, 1950).

A more profound understanding of the above proof did not emerge until much later. If we choose $f(y) = e^{-\pi y^2}$ in the Poisson summation formula, then since $\hat{f}(y) = f(y)$, one observes that for $y > 0$,

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 y} = \frac{1}{\sqrt{y}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 / y}.$$ 

This identity is at the heart of the functional equation of the Riemann zeta function, and is a known transformation formula for Jacobi’s theta function

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z},$$
1.1 Action of a group on a topological space

where \( z = x + iy \) with \( x \in \mathbb{R} \) and \( y > 0 \). If \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is a matrix with integer coefficients \( a, b, c, d \) satisfying \( ad - bc = 1, c \equiv 0 \pmod{4}, c \neq 0 \), then the Poisson summation formula can be used to obtain the more general transformation formula (Shimura, 1973)

\[
\theta \left( \frac{az+b}{cz+d} \right) = \epsilon^{-1}_d \chi_c(d)(cz+d)^{\frac{1}{2}}\theta(z).
\]

Here \( \chi_c \) is the primitive character of order \( \leq 2 \) corresponding to the field extension \( \mathbb{Q}(c^{\frac{1}{2}})/\mathbb{Q} \),

\[
\epsilon_d = \begin{cases} 
1 & \text{if } d \equiv 1 \pmod{4} \\
 i & \text{if } d \equiv -1 \pmod{4},
\end{cases}
\]

and \( (cz+d)^{\frac{1}{2}} \) is the “principal determination” of the square root of \( cz + d \), i.e., the one whose real part is \( > 0 \).

It is now well understood that underlying the functional equation of the Riemann zeta function are the above transformation formulae for \( \theta(z) \). These transformation formulae are induced from the action of a group of matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) on the upper half-plane \( \mathbb{H} = \{ x + iy \mid x \in \mathbb{R}, y > 0 \} \) given by

\[
z \mapsto \frac{az+b}{cz+d}.
\]

The concept of a group acting on a topological space appears to be absolutely fundamental in analytic number theory and should be the starting point for any serious investigations.

1.1 Action of a group on a topological space

Definition 1.1.1 Given a topological space \( X \) and a group \( G \), we say that \( G \) acts continuously on \( X \) (on the left) if there exists a map \( \circ : G \to \text{Func}(X \to X) \) (functions from \( X \) to \( X \)), \( g \mapsto g \circ x \) which satisfies:

- \( x \mapsto g \circ x \) is a continuous function of \( x \) for all \( g \in G \);
- \( g \circ (g' \circ x) = (g \cdot g') \circ x \), for all \( g, g' \in G, x \in X \) where \( \cdot \) denotes the internal operation in the group \( G \);
- \( e \circ x = x \), for all \( x \in X \) and \( e = \text{identity element in } G \).

Example 1.1.2 Let \( G \) denote the additive group of integers \( \mathbb{Z} \). Then it is easy to verify that the group \( \mathbb{Z} \) acts continuously on the real numbers \( \mathbb{R} \) with group
Discrete group actions

action \circ defined by

\[ n \circ x := n + x, \]

for all \( n \in \mathbb{Z}, x \in \mathbb{R}. \) In this case \( e = 0. \)

**Example 1.1.3** Let \( G = GL(2, \mathbb{R})^+ \) denote the group of \( 2 \times 2 \) matrices
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
with \( a, b, c, d \in \mathbb{R} \) and determinant \( ad - bc > 0. \) Let
\[
\mathfrak{h} := \{ x + iy \mid x \in \mathbb{R}, \ y > 0 \}
\]
denote the upper half-plane. For \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})^+ \) and \( z \in \mathfrak{h} \) define:
\[
g \circ z := \frac{az + b}{cz + d}.
\]

Since
\[
\frac{az + b}{cz + d} = \frac{ac|z|^2 + (ad + bc)x + bd}{|cz + d|^2} + i \cdot \frac{(ad - bc) \cdot y}{|cz + d|^2}
\]
it immediately follows that \( g \circ z \in \mathfrak{h}. \) We leave as an exercise to the reader, the verification that \( \circ \) satisfies the additional axioms of a continuous action. One usually extends this action to the larger space \( \mathfrak{h}^* = \mathfrak{h} \cup \{\infty\} \), by defining
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \infty = \begin{cases} 
\frac{a}{c} & \text{if } c \neq 0, \\
\infty & \text{if } c = 0.
\end{cases}
\]

Assume that a group \( G \) acts continuously on a topological space \( X. \) Two elements \( x_1, x_2 \in X \) are said to be equivalent (mod \( G \)) if there exists \( g \in G \) such that \( x_2 = g \circ x_1. \) We define
\[
Gx := \{ g \circ x \mid g \in G \}
\]
to be the equivalence class or orbit of \( x, \) and let \( G \backslash X \) denote the set of equivalence classes.

**Definition 1.1.4** Let a group \( G \) act continuously on a topological space \( X. \) We say a subset \( \Gamma \subset G \) is **discrete** if for any two compact subsets \( A, B \subset X, \) there are only finitely many \( g \in \Gamma \) such that \( (g \circ A) \cap B \neq \emptyset, \) where \( \emptyset \) denotes the empty set.
1.1 Action of a group on a topological space

Example 1.1.5 The discrete subgroup $SL(2, \mathbb{Z})$. Let

$$\Gamma = SL(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\},$$

and let

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\}$$

be the subgroup of $\Gamma$ which fixes $\infty$. Note that $\Gamma_\infty \setminus \Gamma$ is just a set of coset representatives of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where for each pair of relatively prime integers $(c, d) = 1$ we choose a unique $a, b$ satisfying $ad - bc = 1$. This follows immediately from the identity

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + mc & b + md \\ c & d \end{pmatrix}.$$ 

The fact that $SL(2, \mathbb{Z})$ is discrete will be deduced from the following lemma.

Lemma 1.1.6 Fix real numbers $0 < r, 0 < \delta < 1$. Let $R_{r, \delta}$ denote the rectangle

$$R_{r, \delta} = \left\{ x + iy \mid -r \leq x \leq r, \ 0 < \delta \leq y \leq \delta^{-1} \right\}.$$ 

Then for every $\epsilon > 0$, and any fixed set $S$ of coset representatives for $\Gamma_\infty \setminus SL(2, \mathbb{Z})$, there are at most $4 + (4(r + 1)/\epsilon)\delta$ elements $g \in S$ such that $\text{Im}(g \circ z) > \epsilon$ holds for some $z \in R_{r, \delta}$.

Proof Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then for $z \in R_{r, \delta},$

$$\text{Im}(g \circ z) = \frac{y}{c^2y^2 + (cx + d)^2} < \epsilon$$

if $|c| > (y\epsilon)^{-\frac{1}{2}}$. On the other hand, for $|c| \leq (y\epsilon)^{-\frac{1}{2}} \leq (\delta\epsilon)^{-\frac{1}{2}}$, we have

$$\frac{y}{(cx + d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \geq |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently, $\text{Im}(g \circ z) > \epsilon$ only if

$$|c| \leq (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \leq (\epsilon\delta)^{-\frac{1}{2}}(r + 1),$$

and the total number of such pairs (not counting $(c, d) = (0, \pm 1), (\pm 1, 0)$) is at most $4(\epsilon\delta)^{-1}(r + 1)$. \qed
Discrete group actions

It follows from Lemma 1.1.6 that $\Gamma = SL(2, \mathbb{Z})$ is a discrete subgroup of $SL(2, \mathbb{R})$. This is because:

1. It is enough to show that for any compact subset $A \subset \mathfrak{h}$ there are only finitely many $g \in SL(2, \mathbb{Z})$ such that $(g \circ A) \cap A \neq \phi$;
2. Every compact subset of $A \subset \mathfrak{h}$ is contained in a rectangle $R_{r, \delta}$ for some $r > 0$ and $0 < \delta < \delta^{-1}$;
3. $((ag) \circ R_{r, \delta}) \cap R_{r, \delta} = \phi$, except for finitely many $a \in \Gamma_\infty$, $g \in \Gamma_\infty \setminus \Gamma$.

To prove (3), note that Lemma 1.1.6 implies that $(g \circ R_{r, \delta}) \cap R_{r, \delta} = \phi$ except for finitely many $g \in \Gamma_\infty \setminus \Gamma$. Let $S \subset \Gamma_\infty \setminus \Gamma$ denote this finite set of such elements $g$. If $g \notin S$, then Lemma 1.1.6 tells us that it is because $Im(gz) < \delta$ for all $z \in R_{r, \delta}$. Since $Im(agt) = Im(gz)$ for $a \in \Gamma_\infty$, it is enough to show that for each $g \in S$, there are only finitely many $a \in \Gamma_\infty$ such that $((ag) \circ R_{r, \delta}) \cap R_{r, \delta} \neq \phi$. This last statement follows from the fact that $g \circ R_{r, \delta}$ itself lies in some other rectangle $R_{r', \delta'}$, and every $a \in \Gamma_\infty$ is of the form $a = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ ($m \in \mathbb{Z}$), so that

$$\alpha \circ R_{r', \delta'} = \{ x + iy \ | \ -r' + m \leq x \leq r' + m, \ 0 < \delta' \leq \delta'^{-1} \},$$

which implies $(\alpha \circ R_{r', \delta'}) \cap R_{r, \delta} = \phi$ for $|m|$ sufficiently large.

**Definition 1.1.7** Suppose the group $G$ acts continuously on a connected topological space $X$. A fundamental domain for $G \setminus X$ is a connected region $D \subset X$ such that every $x \in X$ is equivalent (mod $G$) to a point in $D$ and such that no two points in $D$ are equivalent to each other.

**Example 1.1.8** A fundamental domain for the action of $\mathbb{Z}$ on $\mathbb{R}$ of Example 1.1.2 is given by

$$\mathbb{Z} \setminus \mathbb{R} = \{ 0 \leq x < 1 \ | \ x \in \mathbb{R} \}.$$ 

The proof of this is left as an easy exercise for the reader.

**Example 1.1.9** A fundamental domain for $SL(2, \mathbb{Z}) \setminus \mathfrak{h}$ can be given as the region $\mathcal{D} \subset \mathfrak{h}$ where

$$\mathcal{D} = \left\{ z \ | \ -\frac{1}{2} \leq \text{Re}(z) \leq \frac{1}{2}, \ |z| \geq 1 \right\},$$

with congruent boundary points symmetric with respect to the imaginary axis.
1.1 Action of a group on a topological space

Note that the vertical line \( V' := \{-\frac{1}{2} + iy \mid y \geq \frac{\sqrt{3}}{2}\} \) is equivalent to the vertical line \( V := \{\frac{1}{2} + iy \mid y \geq \frac{\sqrt{3}}{2}\} \) under the transformation \( z \mapsto z + 1 \).

Furthermore, the arc \( A' := \{z \mid -\frac{1}{2} \leq \Re(z) < 0, \ |z| = 1\} \) is equivalent to the reflected arc \( A := \{z \mid 0 < \Re(z) \leq \frac{1}{2}, \ |z| = 1\} \), under the transformation \( z \mapsto -\frac{1}{z} \).

To show that \( D \) is a fundamental domain, we must prove:

1. For any \( z \in h \), there exists \( g \in SL(2, \mathbb{Z}) \) such that \( g \circ z \in D \);
2. If two distinct points \( z, z' \in D \) are congruent (mod \( SL(2, \mathbb{Z}) \)) then \( \Re(z) = \pm \frac{1}{2} \) and \( z' = z \pm 1 \), or \( |z| = 1 \) and \( z' = -\frac{1}{z} \).

We first prove (1). Fix \( z \in h \). It follows from Lemma 1.1.6 that for every \( \epsilon > 0 \), there are at most finitely many \( g \in SL(2, \mathbb{Z}) \) such that \( g \circ z \) lies in the strip

\[
D_\epsilon := \left\{ w \mid -\frac{1}{2} \leq \Re(w) \leq \frac{1}{2}, \ \epsilon \leq \Im(w) \right\}.
\]

Let \( B_\epsilon \) denote the finite set of such \( g \in SL(2, \mathbb{Z}) \). Clearly, for sufficiently small \( \epsilon \), the set \( B_\epsilon \) contains at least one element. We will show that there is at least one \( g \in B_\epsilon \) such that \( g \circ z \in D \). Among these finitely many \( g \in B_\epsilon \), choose one such that \( \Im(g \circ z) \) is maximal in \( D_\epsilon \). If \( |g \circ z| < 1 \), then for \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \),
Discrete group actions

\[ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and any } m \in \mathbb{Z}, \]

\[ \text{Im}(T^m S g \circ z) = \text{Im} \left( \frac{-1}{g \circ z} \right) = \frac{\text{Im}(g \circ z)}{|g \circ z|^2} > \text{Im}(g \circ z). \]

This is a contradiction because we can always choose \( m \) so that \( T^m S g \circ z \in \mathcal{D}_1 \). So in fact, \( g \circ z \) must be in \( \mathcal{D} \).

To complete the verification that \( \mathcal{D} \) is a fundamental domain, it only remains to prove the assertion (2). Let \( z \in \mathcal{D} \), \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \), and assume that \( g \circ z \in \mathcal{D} \). Without loss of generality, we may assume that

\[ \text{Im}(g \circ z) = \frac{y}{|cz + d|^2} \geq \text{Im}(z), \]

(otherwise just interchange \( z \) and \( g \circ z \) and use \( g^{-1} \)). This implies that \( |cz + d| \leq 1 \) which implies that \( 1 \geq |cz| \geq \frac{y^2}{|d|^2} \). This is clearly impossible if \( |c| \geq 2 \). So we only have to consider the cases \( c = 0, \pm 1 \). If \( c = 0 \) then \( d = \pm 1 \) and \( g \) is a translation by \( b \). Since \( -\frac{1}{2} \leq \text{Re}(z), \text{Re}(g \circ z) \leq \frac{1}{2} \), this implies that either \( b = 0 \) and \( z = g \circ z \) or else \( b = \pm 1 \) and \( \text{Re}(z) = \pm \frac{1}{2} \) while \( \text{Re}(g \circ z) = \frac{1}{2} \). If \( c = 1 \), then \( |z + d| \leq 1 \) implies that \( d = 0 \) unless \( z = e^{2\pi i/3} \) and \( d = 0, 1 \) or \( z = e^{\pi i/3} \) and \( d = 0, -1 \). The case \( d = 0 \) implies that \( |z| \leq 1 \) which implies \( |z| = 1 \). Also, in this case, \( c = 1, d = 0 \), we must have \( b = -1 \) because \( ad - bc = 1 \). Then \( g \circ z = a - 1 \). It follows that \( a = 0 \). If \( z = e^{2\pi i/3} \) and \( d = 1 \), then we must have \( a - b = 1 \). It follows that \( g \circ e^{2\pi i/3} = a - \frac{1}{1 + e^{2\pi i/3}} = a + e^{2\pi i/3} \), which implies that \( a = 0 \) or \( 1 \). A similar argument holds when \( z = e^{\pi i/3} \) and \( d = -1 \). Finally, the case \( c = -1 \) can be reduced to the previous case \( c = 1 \) by reversing the signs of \( a, b, c, d \).

### 1.2 Iwasawa decomposition

This monograph focuses on the general linear group \( GL(n, \mathbb{R}) \) with \( n \geq 2 \). This is the multiplicative group of all \( n \times n \) matrices with coefficients in \( \mathbb{R} \) and non-zero determinant. We will show that every matrix in \( GL(n, \mathbb{R}) \) can be written as an upper triangular matrix times an orthogonal matrix. This is called the Iwasawa decomposition (Iwasawa, 1949).

The Iwasawa decomposition, in the special case of \( GL(2, \mathbb{R}) \), states that every \( g \in GL(2, \mathbb{R}) \) can be written in the form:

\[ g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & \beta \\ y & \delta \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \]

(1.2.1)
where $y > 0$, $x, d \in \mathbb{R}$ with $d \neq 0$ and
\[
\begin{pmatrix}
\alpha & \beta \\
y & \delta
\end{pmatrix} \in O(2, \mathbb{R}),
\]
where
\[
O(n, \mathbb{R}) = \{ g \in GL(n, \mathbb{R}) \mid g \cdot \hat{g} = I \}
\]
is the orthogonal group. Here $I$ denotes the identity matrix on $GL(n, \mathbb{R})$ and $\hat{g}$ denotes the transpose of the matrix $g$. The matrix $\begin{pmatrix} y & x \\
0 & 1 \end{pmatrix}$ in the decomposition (1.2.1) is actually uniquely determined. Furthermore, the matrices $\begin{pmatrix} \alpha & \beta \\
y & \delta \end{pmatrix}$ and $\begin{pmatrix} d & 0 \\
0 & d \end{pmatrix}$ are uniquely determined up to multiplication by $\begin{pmatrix} \pm 1 & 0 \\
0 & \pm 1 \end{pmatrix}$.

Note that explicitly,
\[
O(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos t & -\sin t \\
\pm \sin t & \cos t \end{pmatrix} \mid 0 \leq t \leq 2\pi \right\}.
\]

We shall shortly give a detailed proof of (1.2.1) for $GL(n, \mathbb{R})$ with $n \geq 2$.

The decomposition (1.2.1) allows us to realize the upper half-plane
\[
\mathfrak{h} = \{ x + iy \mid x \in \mathbb{R}, y > 0 \}
\]
as the set of two by two matrices of type
\[
\left\{ \begin{pmatrix} y & x \\
0 & 1 \end{pmatrix} \mid x \in \mathbb{R}, y > 0 \right\},
\]
or by the isomorphism
\[
\mathfrak{h} \equiv GL(2, \mathbb{R})/\langle O(2, \mathbb{R}), Z_2 \rangle, \quad (1.2.2)
\]
where
\[
Z_n = \left\{ \begin{pmatrix} d & 0 \\
\ddots & \ddots \\
0 & d \end{pmatrix} \mid d \in \mathbb{R}, d \neq 0 \right\}
\]
is the center of $GL(n, \mathbb{R})$, and $\langle O(2, \mathbb{R}), Z_2 \rangle$ denotes the group generated by $O(2, \mathbb{R})$ and $Z_2$.

The isomorphism (1.2.2) is the starting point for generalizing the classical theory of modular forms on $GL(2, \mathbb{R})$ to $GL(n, \mathbb{R})$ with $n > 2$. Accordingly, we define the generalized upper half-plane $\mathfrak{h}^n$ associated to $GL(n, \mathbb{R})$. 
Definition 1.2.3 Let $n \geq 2$. The generalized upper half-plane $\mathfrak{h}^n$ associated to $GL(n, \mathbb{R})$ is defined to be the set of all $n \times n$ matrices of the form $z = x \cdot y$ where

$$x = \begin{pmatrix}
1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\
1 & x_{2,3} & \cdots & x_{2,n} \\
\ddots & \ddots & \ddots & \ddots \\
1 & x_{n-1,n} & 1 & \cdots & x_{n-1,n}
\end{pmatrix},
$$

$$y = \begin{pmatrix}
y_{n-1}' \\
y_{n-2}' \\
\vdots \\
y_1'
\end{pmatrix},$$

with $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y_i' > 0$ for $1 \leq i \leq n - 1$.

To simplify later formulae and notation in this book, we will always express $y$ in the form:

$$y = \begin{pmatrix}
y_1 & y_2 & \cdots & y_{n-1} \\
y_2 & y_3 & \cdots & y_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
y_{n-1} & y_1 & 1
\end{pmatrix},$$

with $y_i > 0$ for $1 \leq i \leq n - 1$. Note that this can always be done since $y_i' \neq 0$ for $1 \leq i \leq n - 1$.

Explicitly, $x$ is an upper triangular matrix with $1$s on the diagonal and $y$ is a diagonal matrix beginning with a $1$ in the lowest right entry. Note that $x$ is parameterized by $n \cdot (n - 1)/2$ real variables $x_{i,j}$ and $y$ is parameterized by $n - 1$ positive real variables $y_i$.

Example 1.2.4 The generalized upper half plane $\mathfrak{h}^3$ is the set of all matrices $z = x \cdot y$ with

$$x = \begin{pmatrix}
1 & x_{1,2} & x_{1,3} \\
0 & 1 & x_{2,3} \\
0 & 0 & 1
\end{pmatrix},
$$

$$y = \begin{pmatrix}
y_1 & y_2 & 0 & 0 \\
0 & y_1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},$$

where $x_{1,2}, x_{1,3}, x_{2,3} \in \mathbb{R}$, $y_1, y_2 > 0$. Explicitly, every $z \in \mathfrak{h}^3$ can be written in the form

$$z = \begin{pmatrix}
y_1 & y_2 & x_{1,2} & x_{1,3} \\
0 & y_1 & x_{2,1} & x_{2,3} \\
0 & 0 & 1
\end{pmatrix}.$$