Part I
Relativistic foundations

I know that I am mortal, and the creature of a day . . .
but when I search out the massed wheeling circles of the stars, my feet no longer touch the earth: side by side with Zeus himself, I drink my fill of ambrosia, food of the gods . . .

Claudius Ptolemy, *Mathematical Syntaxis*

1
General ideas and heuristic picture

The aim of this chapter is to introduce the general ideas on which this book is based and to present the picture of quantum spacetime that emerges from loop quantum gravity, in a heuristic and intuitive manner. The style of the chapter is therefore conversational, with little regard for precision and completeness. In the course of the book the ideas and notions introduced here will be made precise, and the claims will be justified and formally derived.

1.1 The problem of quantum gravity

1.1.1 Unfinished revolution

Quantum mechanics (QM) and general relativity (GR) have greatly widened our understanding of the physical world. A large part of the physics of the last century has been a triumphant march of exploration of new worlds opened up by these two theories. QM led to atomic physics, nuclear physics, particle physics, condensed matter physics, semiconductors, lasers, computers, quantum optics . . . GR led to relativistic astrophysics, cosmology, GPS technology . . . and is today leading us, hopefully, towards gravitational wave astronomy.

But QM and GR have destroyed the coherent picture of the world provided by prerelativistic classical physics: each was formulated in terms of assumptions contradicted by the other theory. QM was formulated using an external time variable (the t of the Schrödinger equation) or a fixed, nondynamical background spacetime (the spacetime on which quantum field theory is defined). But this external time variable and this fixed background spacetime are incompatible with GR. In turn, GR was formulated in terms of riemannian geometry, assuming that the metric is a smooth and deterministic dynamical field. But QM requires that any dynamical field be quantized: at small scales it manifests itself in discrete quanta and is governed by probabilistic laws.

We have learned from GR that spacetime is dynamical and we have learned from QM that any dynamical entity is made up of quanta and can be in probabilistic superposition states. Therefore at small scales there should be quanta of space and quanta of time, and quantum superposition of spaces. But what does this mean? We live in a spacetime with quantum properties: a *quantum spacetime*. What is quantum spacetime? How can we describe it?

Classical prerelativistic physics provided a coherent picture of the physical world. This was based on clear notions such as *time, space, matter, particle, wave, force, measurement, deterministic law, . . .* This picture has partially evolved (in particular with the advent of field theory and special relativity) but it has remained consistent and quite stable for three centuries. GR and QM have modified these basic notions in depth. GR has modified the notions of space and time; QM the notions of causality, matter, and measurement. The novel, modified notions do not fit together easily. The new coherent picture is not yet available. With all their immense empirical success, GR and QM have left us with an understanding of the physical world which is unclear and badly fragmented. At the foundations of physics there is today confusion and incoherence.

We want to combine what we have learnt about our world from the two theories and to find a new synthesis. This is a major challenge – perhaps the major challenge – in today's fundamental physics. GR and QM have opened a revolution. The revolution is not yet complete.

With notable exceptions (Dirac, Feynman, Weinberg, DeWitt, Wheeler, Penrose, Hawking, 't Hooft, among others) most of the physicists of the second half of the last century have ignored this challenge. The urgency was to apply the two theories to larger and larger domains. The developments were momentous and the dominant attitude was pragmatic. Applying the new theories was more important than understanding them. But an overly pragmatic attitude is not productive in the long run. Towards the end of the twentieth century, the attention of theoretical physics has been increasingly focusing on the challenge of merging the conceptual novelties of QM and GR.

This book is the account of an effort to do so.

1.1.2 How to search for quantum gravity?

How to search for this new synthesis? Conventional field quantization methods are based on the weak-field perturbation expansion. Their application to GR fails because it yields a nonrenormalizable theory. Perhaps this is not surprising: GR has changed the notions of space and time too radically to docilely agree with flat space quantum field theory. Something else is needed.

In science there are no secure recipes for discovery and it is important to explore different directions at the same time. Currently, a quantum theory of gravity is sought along various paths. The two most developed are loop quantum gravity, described in this book, and string theory. Other research directions include dynamical triangulations, noncommutative geometry, Hartle's quantum mechanics of spacetime (this is not really a specific quantum theory of gravity, but rather a general theoretical framework for general-relativistic quantum theory), Hawking's euclidean sum over geometries, quantum Regge calculus, Penrose's twistor theory, Sorkin's causal sets, 't Hooft's deterministic approach and Finkelstein's theory. The reader can find ample references in the general introductions to quantum gravity mentioned in the note at the end of this chapter. Here, I sketch only the general ideas that motivate the approach described in this book, plus a brief comment on string theory, which is currently the most popular alternative to loop gravity.

Our present knowledge of the basic structure of the physical universe is summarized by GR, quantum theory and quantum field theory (QFT), together with the particle-physics standard model. This set of fundamental theories is inconsistent. But it is characterized by an extraordinary empirical success, nearly unique in the history of science. Indeed, currently there is no evidence of any observed phenomenon that clearly escapes, questions or contradicts this set of theories (or a minor modification of the same, to account, say, for a neutrino mass or a cosmological constant). This set of theories becomes meaningless in certain physical regimes. In these regimes, we expect the predictions of quantum gravity to become relevant and to differ from the predictions of GR and the standard model. These regimes are outside our experimental or observational reach, at least so far. Therefore, we have no direct empirical guidance for searching for quantum gravity – as, say, atomic spectra guided the discovery of quantum theory.

Since quantum gravity is a theory expected to describe regimes that are so far inaccessible, one might worry that anything could happen in these regimes, at scales far removed from our experience. Maybe the search is impossible because the range of the possible theories is too large. This worry is unjustified. If this was the problem, we would have plenty of complete, predictive and coherent theories of quantum gravity. Instead, the situation is precisely the opposite: we haven't any. The fact is that we do have plenty of information about quantum gravity, because we have QM and we have GR. Consistency with QM and GR is an extremely strict constraint.

A view is sometime expressed that some totally new, radical and wild hypothesis is needed for quantum gravity. I do not think that this is the case. Wild ideas pulled out of the blue sky have never made science advance. The radical hypotheses that physics has successfully adopted have always been reluctantly adopted because they were forced upon us by new empirical data – Kepler's ellipses, Bohr's quantization, . . . – or by stringent theoretical deductions – Maxwell's inductive current, Einstein's relativity . . . (see Appendix C). Generally, arbitrary novel hypotheses lead nowhere.

In fact, today we are precisely in one of the typical situations in which theoretical physics has worked at its best in the past. Many of the most striking advances in theoretical physics have derived from the effort of finding a common theoretical framework for two basic and apparently conflicting discoveries. For instance, the aim of combining the keplerian orbits with galilean physics led to newtonian mechanics; combining Maxwell theory with galilean relativity led to special relativity; combining special relativity and nonrelativistic quantum theory led to the theoretical discovery of antiparticles; combining special relativity with newtonian gravity led to general relativity, and so on. In all these cases, major advances have been obtained by "taking seriously"[1] apparently conflicting theories, and exploring the implications of holding the key tenets of both theories for true. Today we are precisely in one of these characteristic situations. We have learned two new very general "facts" about Nature, expressed by QM and GR: we have "just" to figure out what they imply, taken together. Therefore, the question we have to ask is: what have we really learned about the world from QM and from GR? Can we combine these insights into a coherent picture? What we need is a conceptual scheme in which the insights obtained with GR and QM fit together.

This view is *not* the majority view in theoretical physics, at present. There is consensus that QM has been a conceptual revolution, but many do not view GR in the same way. According to many, the discovery of GR has been just the writing of one more field theory. This field theory is, furthermore, likely to be only an approximation to a theory we do not yet know. According to this opinion, GR should not be taken too seriously as a guidance for theoretical developments.

I think that this opinion derives from a confusion: the confusion between the specific form of the Einstein–Hilbert action and the modification of the notions of space and time engendered by GR. The Einstein–Hilbert action might very well be a low-energy approximation of a high-energy theory. But the modification of the notions of space and time does not depend on the specific form of the Einstein–Hilbert action. It depends on its diffeomorphism invariance and its background independence. These properties

[1] In [20], Gell-Mann says that the main lesson to be learnt from Einstein is "to 'take very seriously' ideas that work, and see if they can be usefully carried much further than the original proponent suggested."

(which are briefly illustrated in Section 1.1.3 below, and discussed in detail in Chapter 2) are most likely to hold in the high-energy theory as well. One should not confuse the details of the dynamics of GR with the modifications of the notions of space and time that GR has determined. If we make this confusion, we underestimate the radical novelty of the physical content of GR. The challenge of quantum gravity is precisely to fully incorporate this radical novelty into QFT. In other words, the task is to understand what is a general-relativistic QFT, or a background-independent QFT.

Today many physicists prefer disregarding or postponing these foundational issues and, instead, choose to develop and adjust current theories. The most popular strategy towards quantum gravity, in particular, is to pursue the line of research grown in the wake of the success of the standard model of particle physics. The failure of perturbative quantum GR is interpreted as a replay of the failure of Fermi theory.[2] Namely, as an indication that we must modify GR at high energy. With the input of the grand-unified-theories (GUTs), supersymmetry, and the Kaluza–Klein theory, the search for a high-energy correction of GR free from bad ultraviolet divergences has led to higher derivative theories, supergravity, and finally to string theory.

Sometimes the claim is made that the quantum theory of gravity has already been found and it is string theory. Since this is a book about quantum gravity without strings, I should say a few words about this claim. String theory is based on a physical hypothesis: elementary objects are extended, rather than particle-like. This hypothesis leads to a very rich unified theory, which contains much phenomenology, including (with suitable inputs) fermions, Yang–Mills fields and gravitons, and is expected by many to be free of ultraviolet divergences. The price to pay for these theoretical results is a gigantic baggage of additional physics: supersymmetry, extra dimensions, an infinite number of fields with arbitrary masses and spins, and so on.

So far, nothing of this new physics shows up in experiments. Supersymmetry, in particular, has been claimed to be on the verge of being discovered for years, but hasn't shown up. Unfortunately, so far the theory can accommodate any disappointing experimental result because it is hard to derive precise new quantitative physical predictions, with which the theory could be falsified, from the monumental mathematical apparatus of the theory. Furthermore, even recovering the real world is not easy within the theory: the search for a compactification leading to the

[2]Fermi theory was an empirically successful but nonrenormalizable theory of the weak interactions, just as GR is an empirically successful but nonrenormalizable theory of the gravitational interaction. The solution has been the Glashow–Weinberg–Salam electroweak theory, which corrects Fermi theory at high energy.

standard model, with its families and masses and no instabilities, has not yet succeeded, as far as I know. It is clear that string theory is a very interesting hypothesis, but certainly not an established theory. It is therefore important to pursue alternative directions as well.

String theory is a direct development of the standard model and is deeply rooted in the techniques and the conceptual framework of flat space QFT. As I shall discuss in detail throughout this book, many of the tools used in this framework – energy, unitary time evolution, vacuum state, Poincaré invariance, S-matrix, objects moving in a spacetime, Fourier transform, . . . – no longer make sense in the quantum gravitational regime, in which the gravitational field cannot be approximated by a background spacetime – perhaps not even asymptotically.[3] Therefore string theory does not address directly the main challenge of quantum gravity: understanding what is a background-independent QFT. Facing this challenge directly, before worrying about unification, leads, instead, to the direction of research investigated by loop quantum gravity.[4]

The alternative to the line of research followed by string theory is given by the possibility that the failure of perturbative quantum GR is *not* a replay of Fermi theory. That is, it is not due to a flaw of the GR action, but, instead, it is due to the fact that the conventional weak-field quantum perturbation expansion cannot be applied to the gravitational field.

This possibility is strongly supported a posteriori by the results of loop quantum gravity. As we shall see, loop quantum gravity leads to a picture of the short-scale structure of spacetime extremely different from that of a smooth background geometry. (There are hints in this direction from string theory calculations as well [23].) Spacetime turns out to have a nonperturbative, quantized, discrete structure at the Planck scale, which is explicitly described by the theory. The ultraviolet divergences are cured by this structure. The ultraviolet divergences that appear in the perturbation expansion of conventional QFT are a consequence of the fact that

[3] To be sure, the development of string theory has incorporated many aspects of GR, such as curved spacetimes, horizons, black holes and relations between different backgrounds. But this is far from a background-independent framework, such as the one realized by GR in the classical context. GR is not about physics on a curved spacetime, or about relations between different backgrounds: it is about the dynamics of spacetime. A background-independent fundamental definition of string theory is being actively searched for along several directions, but so far the definition of the theory and all calculations rely on background metric spaces.

[4] It has been repeatedly suggested that loop gravity and string theory might merge, because loop gravity has developed precisely the background-independent QFT methods that string theory needs [21]. Also, excitations over a weave (see Section 6.7.1) have a natural string structure in loop gravity [22].

we erroneously replace this discrete Planck-scale structure with a smooth background geometry.

If this is physically correct, ultraviolet divergences do not require the heavy machinery of string theory to be cured. On the other hand, the conventional weak-field perturbative methods cannot be applied, because we cannot work with a fixed smooth background geometry. We must therefore adapt QFT to the full conceptual novelty of GR, and in particular to the change in the notion of space and time induced by GR. What are these changes? I sketch an answer below, leaving a complete discussion to Chapter 2.

1.1.3 The physical meaning of general relativity

GR is the discovery that spacetime and the gravitational field are the same entity. What we call "spacetime" is itself a physical object, in many respects similar to the electromagnetic field. We can say that GR is the discovery that there is no spacetime at all. What Newton called "space", and Minkowski called "spacetime," is unmasked: it is nothing but a dynamical object – the gravitational field – in a regime in which we neglect its dynamics.

In newtonian and special-relativistic physics, if we take away the dynamical entities – particles and fields – what remains is space and time. In general-relativistic physics, if we take away the dynamical entities, nothing remains. The space and time of Newton and Minkowski are re-interpreted as a configuration of one of the fields, the gravitational field. This implies that physical entities – particles and fields – are not immersed in space, and moving in time. They do not live on spacetime. They live, so to say, on one another.

It is as if we had observed in the ocean many animals living on an island: animals on the island. Then we discover that the island itself is in fact a great whale. So the animals are no longer on the island, just animals on animals. Similarly, the Universe is not made up of fields on spacetime; it is made up of fields on fields. This book studies the far-reaching effect that this conceptual shift has on QFT.

One consequence is that the quanta of the field cannot live in spacetime: they must build "spacetime" themselves. This is precisely what the quanta of space do in loop quantum gravity.

We may continue to use the expressions "space" and "time" to indicate aspects of the gravitational field, and I do so in this book. We are used to this in classical GR. But in the quantum theory, where the field has quantized "granular" properties and its dynamics is quantized and therefore only probabilistic, most of the "spatial" and "temporal" features of the gravitational field are lost.

Therefore, to understand the quantum gravitational field we must abandon some of the emphasis on geometry. Geometry represents the classical gravitational field, but not quantum spacetime. This is not a betrayal of Einstein's legacy: on the contrary, it is a step in the direction of "relativity" in the precise sense meant by Einstein. Alain Connes has described beautifully the existence of two points of view on space: the geometric one, centered on space points, and the algebraic, or "spectral" one, centered on the algebra of dual spectral quantities. As emphasized by Alain, quantum theory forces us to a complete shift to this second point of view, because of noncommutativity. In the light of quantum theory, continuous spacetime cannot be anything else than an approximation in which we disregard quantum noncommutativity. In loop gravity, the physical features of space appear as spectral properties of quantum operators that describe our (the observers') interactions with the gravitational field.

The key conceptual difficulty of quantum gravity is therefore to accept the idea that we can do physics in the absence of the familiar stage of space and time. We need to free ourselves from the prejudices associated with the habit of thinking of the world as "inhabiting space" and "evolving in time". Chapter 3 describes a language for describing mechanical systems in this generalized conceptual framework.

This absence of the familiar spacetime "stage" is called the *background independence* of the classical theory. Technically, it is realized by the gauge invariance of the action under (active) diffeomorphisms. A diffeomorphism is a transformation that smoothly drags all dynamical fields and particles from one region of the four-dimensional manifold to another (the precise definition of these transformations is given in Chapter 2). In turn, gauge invariance under diffeomorphism (or *diffeomorphism invariance*) is the consequence of the combination of two properties of the action: its invariance under arbitrary changes of coordinates and the fact that there is no nondynamical "background" field.

1.1.4 Background-independent quantum field theory

Is quantum mechanics[5] compatible with the general-relativistic notions of space and time? It is, provided that we choose a sufficiently general formulation. For instance, the Schrödinger picture is only viable for theories where there is a global observable time variable t; this conflicts with GR, where no such variable exists. Therefore, the Schrödinger picture makes little sense in a background-independent context. But there

[5]I use the expression "quantum mechanics" to indicate the theory of all quantum systems, with a finite or infinite number of degrees of freedom. In this sense QFT is part of quantum mechanics.

are formulations of quantum theory that are more general than the Schrödinger picture. In Chapter 5, I describe a formulation of QM sufficiently general to deal with general-relativistic systems. (For another relativistic formulation of QM, see [24].) Formulations of this kind are sometimes denoted "generalized quantum mechanics." I prefer to use "quantum mechanics" to denote any formulation of quantum theory, irrespective of its generality, just as "classical mechanics" is used to designate formalisms with different degrees of generality, such as Newton's, Lagrange's, Hamilton's or symplectic mechanics.

On the other hand, most of the conventional machinery of perturbative QFT is profoundly incompatible with the general-relativistic framework. There are many reasons for this:

- The conventional formalism of QFT relies on Poincaré invariance. In particular, it relies on the notion of energy and on the existence of the nonvanishing hamiltonian operator that generates unitary time evolution. The vacuum, for instance, is the state that minimizes the energy. Generally, there is no global Poincaré invariance, no general notion of energy and no nonvanishing hamiltonian operator in a general-relativistic theory.
- At the root of conventional QFT is the physical notion of particle. The theoretical experience with QFT on curved spacetime [25] and on the relation between acceleration and temperature in QFT [26] indicates that in a generic gravitational situation the notion of particle can be quite delicate. (This point is discussed in Section 5.3.4.)
- Consider a conventional renormalized QFT. The physical content of the theory can be expressed in terms of its n-point functions $W(x_1, \ldots, x_n)$. The n-point functions reflect the invariances of the classical theory. In a general-relativistic theory, invariance under a coordinate transformation $x \to x' = x'(x)$ implies immediately that the n-point functions must satisfy

$$W(x_1, \ldots, x_n) = W(x'(x_1), \ldots, x'(x_n)) \tag{1.1}$$

  and therefore (if the points in the argument are distinct) it must be a constant! That is,

$$W(x_1, \ldots, x_n) = constant. \tag{1.2}$$

  Clearly, we are immediately in a very different framework from conventional QFT.
- Similarly, the behavior for small $|x - y|$ of the two-point function of a conventional QFT

$$W(x, y) = \frac{constant}{|x - y|^d}, \tag{1.3}$$