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Linear and Projective Representations of Symmetric Groups

ALEXANDER KLESHCHEV
University of Oregon

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Preface

The subject of this book is representation theory of symmetric groups. We explain a new approach to this theory based on the recent work of Lascoux, Leclerc, Thibon, Ariki, Grojnowski, Brundan, Kleshchev, and others. We are mainly interested in *modular* representation theory, although everything works in arbitrary characteristic, and in case of characteristic 0 our approach is somewhat similar to the theory of Okounkov and Vershik [OV], described in Chapter 2 of this book. The methods developed here are quite general and they apply to a number of related objects: finite and affine Iwahori–Hecke algebras of type A, cyclotomic Hecke algebras, spin-symmetric groups, Sergeev algebras, Hecke–Clifford superalgebras, affine and cyclotomic Hecke–Clifford superalgebras, ... We concentrate on symmetric and spin-symmetric groups though.

We now outline some of the ideas which lead to the new approach. Let us concentrate on the modular case, as this is where things get really interesting. So let F be a field of characteristic $p > 0$, and S_n be the symmetric group. Irreducible FS_n-modules were classified by James. His approach is as follows (see [J] for details). Let S^λ be the Specht module corresponding to a partition λ of n (the Specht construction works over any field and even over $\mathbb{Z}$). This module has a canonical S_n-invariant bilinear form. The form is non-zero if and only if λ is *p-regular*, that is no non-zero part of λ is repeated p or more times. In this case the radical of the form, call it Q^λ, is a maximal proper submodule. Thus $D^\lambda := S^\lambda/Q^\lambda$ is an irreducible FS_n-module. Finally, $\{D^\lambda \mid \lambda \text{ is a } p\text{-regular partition of } n\}$ is a complete set of irreducible FS_n-modules.

The main unsolved problem in modular representation theory of symmetric groups is to find (Brauer) characters and in particular dimensions of irreducible modules. These will be known if any of the following two equivalent problems can be solved (see [K_6] for the proof of equivalence). Denote by $\mathcal{P}(n)$ (resp. $\mathcal{P}_p(n)$) the set of all partitions (resp. p-regular partitions) of n.

Decomposition numbers problem. Find the composition multiplicities

$$[S^\lambda : D^\mu] \tag{0.1}$$

for any $\lambda \in \mathcal{P}(n)$ and $\mu \in \mathcal{P}_p(n)$.

Branching problem. Determine the composition multiplicities of the restriction

$$[\mathrm{res}^{S_n}_{S_{n-1}} D^\lambda : D^\mu] \tag{0.2}$$

for any $\lambda \in \mathcal{P}_p(n)$ and $\mu \in \mathcal{P}_p(n-1)$.

Motivated by the second problem, the author [K_1] obtained certain partial *branching rules*, including an explicit description of the socle of the restriction $\mathrm{res}^{S_n}_{S_{n-1}} D^\lambda$, which is equivalent to the description of the spaces

$$\mathrm{Hom}_{FS_{n-1}}(D^\mu, \mathrm{res}^{S_n}_{S_{n-1}} D^\lambda) \qquad \big(\lambda \in \mathcal{P}_p(n),\ \mu \in \mathcal{P}_p(n-1)\big). \tag{0.3}$$

Another result from [K_1] describes the spaces

$$\mathrm{Hom}_{FS_{n-1}}(S^\mu, \mathrm{res}^{S_n}_{S_{n-1}} D^\lambda) \qquad \big(\lambda \in \mathcal{P}_p(n),\ \mu \in \mathcal{P}_p(n-1)\big). \tag{0.4}$$

It turns out that the spaces in (0.3) and (0.4) are at most 1-dimensional, which gives two different generalizations of the multiplicity freeness of the branching rule in characteristic 0. The solution is in terms of delicate combinatorial notions of a *normal node* and a *good node* of a Young diagram (see Chapter 11): the space (0.3) (resp. (0.4)) is non-trivial if and only if μ is obtained from λ by removing a good node (resp. normal node). As observed in [K_2], it follows from this description that all irreducible modules appearing in the socle of $\mathrm{res}^{S_n}_{S_{n-1}} D^\lambda$ belong to different blocks, the fact sometimes referred to as the *strong multiplicity freeness* of the branching rule. A number of further results on the modular branching problem was established in [K_5, K_4, BK_1]. For example, in [K_4] we have described the multiplicity (0.2) when μ is obtained from λ by removing a node. It turns out that this multiplicity is non-zero if and only if the node is normal, but it can be arbitrarily large.

Many applications of the branching rules were obtained, see for example [K_2, FK, BO_1, BK_4]. But the most important consequence was that they led to a discovery of deep connections between modular representation theory and the theory of crystal bases. The link was first made by Lascoux, Leclerc, and Thibon [LLT]. The nodes of Young diagrams come with residues, which are elements of $\mathbb{Z}/p\mathbb{Z}$, and we obtain a structure of a directed colored graph on the set of all p-regular partitions: by definition, there is an arrow from μ to λ of color $i \in \mathbb{Z}/p\mathbb{Z}$ if and only if μ is obtained from λ by removing a good node of residue i. Lascoux, Leclerc, and Thibon made the startling combinatorial observation that this "branching graph" coincides with the crystal graph of the basic representation of the affine Kac–Moody algebra $\mathfrak{g} = A_{p-1}^{(1)}$, determined explicitly by Misra and Miwa [MiMi]. It turned out later that the same observation applies to the branching graph for the associated complex Iwahori–Hecke algebras at a primitive $(\ell+1)$th root of unity and the crystal graph of the basic representation of the affine Kac–Moody algebra $\mathfrak{g} = A_{\ell}^{(1)}$, see [B]. In this latter case, Lascoux, Leclerc, and Thibon conjectured moreover that the coefficients of the canonical basis of the basic representation coincide with the decomposition numbers (0.1) for the Iwahori–Hecke algebras.

This conjecture was proved by Ariki [A_1] (see also Grojnowski [G_1]). More generally, Ariki established a similar result connecting the canonical basis of an arbitrary integrable highest weight module of $\mathfrak{g}$ to the representation theory of the corresponding cyclotomic Hecke algebra, as defined in [Ch, AK, BM]. Note that Ariki's work is concerned with the cyclotomic Hecke algebras over the ground field $\mathbb{C}$, but Ariki and Mathas [A_2, AM] were later able to extend the classification of the irreducible modules, but not the result on decomposition numbers, to arbitrary fields. For further developments related to the LLT conjecture, see [LT_1, VV, Sch, A_3].

Subsequently, Grojnowski and Vazirani [G_2, G_3, GV, V_2] have developed a powerful new approach to (among other things) the classification of the irreducible modules of the cyclotomic Hecke algebras. The approach is valid over an arbitrary ground field, and is entirely independent of the "Specht module theory" that plays an important role in Ariki's work. Branching rules are built in from the outset, resulting in an explanation and generalization of the link between modular branching rules and crystal graphs. The methods are purely algebraic, exploiting affine Hecke algebras in the spirit of [BZ, Z_1] and others. On the other hand, results on decomposition numbers do not follow, since that ultimately depends on the geometric work of Kazhdan, Lusztig, and Ginzburg.

In this book, we explain Grojnowski's approach to the theory in the case of degenerate affine Hecke algebra. In particular, we obtain an algebraic construction purely in terms of the representation theory of degenerate affine Hecke algebras of the positive part $U_{\mathbb{Z}}^{+}$ of the enveloping algebra of $\mathfrak{g} = A_{p-1}^{(1)}$, as well as of Kashiwara's highest weight crystals $B(\infty)$ and $B(\lambda)$ for each dominant weight λ. These emerge as the modular branching graphs of the appropriate algebras. As a consequence, a parametrization of irreducible FS_n-modules, classification of blocks ("Nakayama's conjecture"), and some of the modular branching rules mentioned above follow from the special case $\lambda = \Lambda_0$ of the main results.

Part II of the book deals with representation theory of Schur's double covers of symmetric groups. This is equivalent to studying projective (or spin) representations of symmetric groups. The spin theory in characteristic p was developed in [BK$_3$] (cf. [BMO, ABO$_1$, ABO$_2$, MoY$_2$])–it is parallel to the theory of linear representations described above, but with the role of the Kac–Moody algebra $\mathfrak{g} = A_{p-1}^{(1)}$ played by the twisted Kac–Moody algebra $\mathfrak{g} = A_{p-1}^{(2)}$. We note that the modular irreducible spin representations of S_n were first classified in [BK$_2$], following a conjecture in [LT$_2$], using a more classical approach via "Specht modules". However, that approach did not allow us to obtain any branching rules.

We hope that having both linear and spin theory under one cover will be useful for the reader. The two theories are actually very similar, if developed from the new point of view adopted in this book. In fact, a glance at the contents shows that many sections of Part II are exactly parallel to the corresponding sections of Part I.

Let us now describe the contents of the book in more detail. We note that each chapter has its own introduction where the results of the chapter are motivated and described, sometimes informally. Chapter 1 contains notation and some basic preliminary results. Chapter 2 is a presentation of the beautiful theory of Okounkov and Vershik in characteristic 0. It is not directly related to the rest of the book. However, it is a nice introduction to some ideas employed further on, and might be a good place to start for less-advanced readers. Also, it is perhaps the shortest way to some key results of the classical representation theory of symmetric groups in characteristic 0, such as classification of irreducible representations, Young's formulas, and the Murnaghan–Nakayama formula.

Degenerate affine Hecke algebras $\mathcal{H}_n$ are introduced in Chapter 3. Basis Theorem for $\mathcal{H}_n$ is proved and the center of $\mathcal{H}_n$ is described. We introduce parabolic subalgebras $\mathcal{H}_\mu \subset \mathcal{H}_n$ and the corresponding induction and restriction functors ind_μ^n and res_μ^n. "Mackey Theorem" is a result describing a filtration of $\mathrm{res}_\mu^n \mathrm{ind}_\nu^n$ by certain induced modules. An important result relating induction and duality is proved in Section 3.7.

In Chapter 4, we introduce the formal characters of finite dimensional $\mathcal{H}_n$-modules, discuss central characters and blocks, and then study in detail a remarkable irreducible $\mathcal{H}_n$-module, called Kato module, as well as its "covering modules".

The functors e_a and their versions Δ_a and Δ_{a^m}, which are affine analogues of Robinson's a-restriction functors for symmetric groups, are studied in

Chapter 5. The (affine version of the) strong multiplicity freeness of the branching rule is established in Corollary 5.1.7. This allows us to define the crystal operators $\tilde{e}_a$ and $\tilde{f}_a$ in Section 5.2 as the socle and the head of a certain direct summand of restriction and induction, respectively. We then prove that the formal characters of irreducible modules are linearly independent and use the crystal operators $\tilde{f}_a$ to label the irreducible $\mathcal{H}_n$-modules. Section 5.5 contains some further branching results.

In Chapter 6 we establish a sufficient condition for the irreducibility of the module, induced from an irreducible module over a parabolic subalgebra (this condition is far from necessary–see for example [LNT] for results on the subtle problem of finding necessary and sufficient condition). Then we calculate characters of some irreducible $\mathcal{H}_n$-modules for $n \leq 4$. These calculations provide the reader with concrete examples to play with, but also are important for the theory, as they will imply that the operators e_i on the Grothendieck group satisfy Serre relations. Finally, some higher crystal operators are introduced in Section 6.3–these play only a technical role.

Integral representations and (degenerate) cyclotomic Hecke algebras are treated in Chapter 7. We explain why it is enough to study integral representations and reveal their relations with cyclotomic Hecke algebras $\mathcal{H}_n^\lambda$. We introduce Lie theoretic notation related to the Kac–Moody algebra $\mathfrak{g}$ of type $A_{p-1}^{(1)}$, which will be used until the end of Part I. The main results of the chapter are Basis Theorem for cyclotomic Hecke algebras, Cyclotomic Mackey Theorem, and the fact that τ-duality commutes with induction for cyclotomic Hecke algebras.

In Chapter 8 we study the cyclotomic analogues e_i^λ of the functors e_i and introduce the “dual” functors f_i^λ and related notions. We also introduce the divided powers functors which generalize e_i^λ and f_i^λ. The main goal of the Chapter is to get the f_i^λ-analogues of the results on e_i obtained in Chapter 5, that is to get the results on induction similar to our previous results on restriction. This turns out to be quite a bit harder.

Key Chapter 9 begins by defining a Hopf algebra structure on the Grothendieck group $K(\infty)$ of integral $\mathcal{H}_n$-modules for all $n \geq 0$, operations coming from induction and restriction. We prove in Theorem 9.5.3 that the dual algebra $K(\infty)^*$ can be identified with the universal enveloping algebra of the positive part of the Kac–Moody algebra $\mathfrak{g}$. We also obtain a natural action of $K(\infty)^*$ on the Grothendieck group $K(\lambda)$ of finite dimensional $\mathcal{H}_n^\lambda$-modules for all $n \geq 0$–under this action the Chevalley generators e_i act as operators e_i^λ, which come from representation theory of $\mathcal{H}_n^\lambda$. This action is extended with the operators f_i^λ and h_i^λ to give the action of the full Kac–Moody algebra $\mathfrak{g}$, and the module $K(\lambda)$ is identified with the irreducible high weight module

$V(\lambda)$. Finally, the weight spaces of $V(\lambda)$ are interpreted in terms of blocks of $\mathcal{H}_n^\lambda$-modules, see Section 9.6.

In Chapter 10 we identify the crystal graph of $V(\lambda)$ as the socle branching graph of the cyclotomic Hecke algebras $\mathcal{H}_n^\lambda$. The consequences for symmetric groups are deduced in Chapter 11, where we specialize to the case $\lambda = \Lambda_0$ and use the explicit description of the corresponding crystal graph to label the irreducible modules by p-regular partitions, to describe the blocks, and to deduce some of the branching rules. We also get useful results on formal characters.

Part II starts with Chapter 12, which is a review on superalgebras and their representations. This will be needed for the spin theory, as it turns out extremely convenient to consider the twisted group algebra $\mathcal{T}_n$ of S_n as a superalgebra and then work in the category of $\mathcal{T}_n$-supermodules. In fact, it is even more convenient to work with the Sergeev superalgebra $\mathcal{Y}_n$, which is defined in Chapter 13. We also prove in this Chapter that $\mathcal{Y}_n$ is "almost Morita equivalent" to $\mathcal{T}_n$. Chapters 14–22 are all parallel to the corresponding chapters of Part I, as indicated in the beginning of each of them, so we do not review them here in detail.

I have benefited greatly from collaboration with Jon Brundan. I was greatly influenced by the work of Jantzen and Seitz; Lascoux, Leclerc, and Thibon; Ariki; and, especially, Grojnowski. The idea to write this book appeared when I lectured on the topic at the University of Wisconsin-Madison during my sabbatical leave from Oregon. I am very grateful to Georgia Benkart, Ken Ono, and especially Arun Ram for their hospitality. I should also thank the editors–without their repeated (patient) emails I would never have finished. Most of all, I am indebted to my family (without whom this book might have been written faster, but that's not the point...).