PART I

Linear representations

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Alexander Kleshchev
Excerpt
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Notation and generalities

Throughout the book: $\mathbb{Z}_+$ is the set of non-negative integers and F is an algebraically closed field of characteristic $p \geq 0$. Throughout Part I

$$I := \mathbb{Z} \cdot 1 \subset F. \tag{1.1}$$

If $p = \operatorname{char} F > 0$ then I is identified with $\{0, 1, \dots, p-1\}$, and if $p = 0$ then $I = \mathbb{Z}$.

If $\mathcal{A}$ is an associative F-algebra we denote by $\mathcal{A}$-mod the category of all finite dimensional left $\mathcal{A}$-modules and by $\mathcal{A}$-proj $\subset \mathcal{A}$-mod the full subcategory of all projective $\mathcal{A}$-modules. We write $K(\mathcal{A}\text{-mod}), K(\mathcal{A}\text{-proj})$ for the corresponding Grothendieck groups. The embedding $\mathcal{A}$-proj $\subset \mathcal{A}$-mod induces the natural *Cartan map*

$$\omega : K(\mathcal{A}\text{-proj}) \to K(\mathcal{A}\text{-mod}).$$

Note that in general ω does not have to be injective.

Let $M \in \mathcal{A}$-mod. The *socle* of M, written $\operatorname{soc} M$, is the largest completely reducible submodule of M, and the *head* of M, written $\operatorname{hd} M$, is the largest completely reducible quotient module of M. If V is an irreducible $\mathcal{A}$-module, we write $[M : V]$ for the multiplicity of V as a composition factor of M.

For algebras $\mathcal{A}, \mathcal{B}$, an $\mathcal{A}$-module M, and a $\mathcal{B}$-module N, we write $M \boxtimes N$ for the outer tensor product, that is the tensor product of vector spaces $M \otimes N$ considered as an $\mathcal{A} \otimes \mathcal{B}$-module in the usual way.

If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, and M is a $\mathcal{B}$-module we write $\operatorname{ind}_{\mathcal{B}}^{\mathcal{A}} M$ or $\operatorname{ind}^{\mathcal{A}}$ for the induced module $\mathcal{A} \otimes_{\mathcal{B}} M$. We may consider $\operatorname{ind}_{\mathcal{B}}^{\mathcal{A}}$ as a functor from the category of $\mathcal{B}$-modules to the category of $\mathcal{A}$-modules. This functor is left adjoint to the restriction functor $\operatorname{res}_{\mathcal{B}}^{\mathcal{A}}$ (or $\operatorname{res}_{\mathcal{B}}$) going in the other direction. If $\mathcal{A}$ is free as a right $\mathcal{B}$-module the induction functor is exact.

We denote by $\mathcal{Z}(\mathcal{A})$ the center of $\mathcal{A}$. By a *central character* of A we mean a (unital) algebra homomorphism $\chi : \mathcal{Z}(\mathcal{A}) \to F$. For a central character χ the corresponding *block* is the full subcategory A-mod$[\chi]$ of $\mathcal{A}$-mod consisting of all modules $M \in \mathcal{A}$-mod such that $(z - \chi(z))^k M = 0$ for $k \gg 0$. We have a decomposition

$$\mathcal{A}\text{-mod} = \bigoplus_{\chi} \mathcal{A}\text{-mod}[\chi]$$

as χ runs over all central characters. If $\mathcal{A}$ is finite dimensional, then two non-isomorphic irreducible $\mathcal{A}$-modules L and M belong to the same block if and only if there exists a chain $L \cong L_0, L_1, \dots, L_m \cong M$ of irreducible $\mathcal{A}$-modules with either $\mathrm{Ext}^1_{\mathcal{A}}(L_i, L_{i+1}) \neq 0$ or $\mathrm{Ext}^1_{\mathcal{A}}(L_{i+1}, L_i) \neq 0$ for each i.

Let $\mathcal{B}$ be a subalgebra of an F-algebra $\mathcal{A}$ and $\mathcal{C}$ be the centralizer of $\mathcal{B}$ in $\mathcal{A}$. If V is an $\mathcal{A}$-module and W is a $\mathcal{B}$-module then $\mathrm{Hom}_{\mathcal{B}}(W, \mathrm{res}_{\mathcal{B}} V)$ is naturally a $\mathcal{C}$-module with respect to the action $(cf)(w) = cf(w)$ for $w \in W$, $f \in \mathrm{Hom}_{\mathcal{B}}(W, \mathrm{res}_{\mathcal{B}} V)$, $c \in \mathcal{C}$.

Lemma 1.0.1 *Let $\mathcal{B} \subseteq \mathcal{A}$ be semisimple finite dimensional F-algebras. If V is irreducible over $\mathcal{A}$ and W is irreducible over $\mathcal{B}$ then*

$$\mathrm{Hom}_{\mathcal{B}}(W, \mathrm{res}_{\mathcal{B}} V)$$

is irreducible over $\mathcal{C}$.

Proof By Wedderburn–Artin, we may assume that $\mathcal{A} = \mathrm{End}(V)$. Decompose $\mathrm{res}_{\mathcal{B}} V = W^{\oplus k} \oplus X$, where W is not a composition factor of X. Then the algebra $\mathrm{End}_{\mathcal{B}}(W^{\oplus k})$, naturally contained in $\mathcal{C}$, acts on the space $\mathrm{Hom}_{\mathcal{B}}(W, \mathrm{res}_{\mathcal{B}} V)$ as the full endomorphism algebra. □

For any $n \geq 0$, let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a *partition of* n, that is a non-increasing sequence of non-negative integers summing to n. If $p > 0$, the partition α is called *p-regular* if for any $k > 0$ we have

$$\sharp\{j \mid \alpha_j = k\} < p.$$

By definition, every partition is 0-regular. Let $\mathcal{P}(n)$ (resp. $\mathcal{P}_p(n)$) denote the set of all (resp. all p-regular) partitions of n. Thus $\mathcal{P}(n) = \mathcal{P}_0(n)$. Set

$$\mathcal{P} := \bigcup_{n \geq 0} \mathcal{P}(n) \quad \text{and} \quad \mathcal{P}_p := \bigcup_{n \geq 0} \mathcal{P}_p(n).$$

We identify a partition α with its *Young diagram*

$$\alpha = \{(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \leq \alpha_r\}.$$

Elements $(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ are called *nodes*. We label the nodes of α with *residues*, which are elements of I. By definition, the residue of the node (r, s)

is $s-r \pmod p$ if p is positive, and simply $s-r$ if $p=0$. The residue of the node A is denoted res A. Define the *residue content* of α to be the tuple

$$\mathrm{cont}(\alpha) = (\gamma_i)_{i\in I}, \tag{1.2}$$

where for each $i \in I$, γ_i is the number of nodes of residue i contained in the diagram α.

Let $i \in I$ be some fixed residue. A node $A=(r,s)\in\alpha$ is called *i-removable* (resp. *i-addable*) for α if res $A=i$ and $\alpha_A := \alpha \setminus \{A\}$ (resp. $\alpha^A := \alpha \cup \{A\}$) is a Young diagram of a partition. A node is called *removable* (resp. *addable*) if it is i-removable (resp. i-addable) for some i. Thus, for example, a removable node is always of the form (m, α_m) with $\alpha_m > \alpha_{m+1}$.

Throughout the book, S_n denotes the symmetric group on n letters. The permutations act on numbers $1,\dots,n$ on the *left* so that for the product we have, for example, $(1,2)(2,3)=(1,2,3)$. S_n also acts on n-tuples of objects by place permutations on the right:

$$(a_1,a_2,\dots,a_n)\cdot w = (a_{w1},\dots,a_{wn})$$

or on the left:

$$w\cdot(a_1,a_2,\dots,a_n) = (a_{w^{-1}1},\dots,a_{w^{-1}n}).$$

The length function on S_n in the sense of Coxeter groups is denoted by ℓ. The number $\ell(w)$ can be characterized as the number of inversions in the permutation w.

Finally we recall one classical result. Let

$$\mathcal{P}_n = F[x_1,\dots,x_n]$$

be the polynomial algebra in n indeterminates,

$$\mathcal{Z}_n := F[x_1,\dots,x_n]^{S_n}$$

be the ring of symmetric polynomials, and $\mathcal{Z}_n^+ \subset \mathcal{Z}_n$ be the symmetric polynomials without free term. The following fact is well known over $\mathbb{C}$. That it holds over $\mathbb{Z}$, and hence over F, is proved in [St].

Theorem 1.0.2 *$\mathcal{P}_n$ is a free module of rank $n!$ over $\mathcal{Z}_n$. Moreover we can take the set*

$$B := \{x_1^{a_1}\dots x_n^{a_n} \mid 0 \le a_i < i \text{ for all } 1 \le i \le n\}$$

as a basis. In particular, the cosets of elements of B form a basis of the algebra $\mathcal{P}_n/\mathcal{P}_n\mathcal{Z}_n^+$.

A slightly more general result easily follows:

Corollary 1.0.3 *Let $r \leq n$. Then $\mathcal{P}_n^{S_r}$ is a free module of rank $n!/r!$ over $\mathcal{Z}_n$. Moreover we can take the set*

$$B := \{x_{r+1}^{a_{r+1}} \dots x_n^{a_n} \mid 0 \leq a_i < i \text{ for all } r+1 \leq i \leq n\}$$

as a basis.

Proof It suffices to prove that elements of B generate $\mathcal{P}_n^{S_r}$ as a module over $\mathcal{Z}_n$. Let $f \in \mathcal{P}_n^{S_r}$. In view of Theorem 1.0.2, we can write

$$f = \sum f_{\underline{a}} x_1^{a_1} \dots x_n^{a_n}, \tag{1.3}$$

where the summation is over all n-tuples $\underline{a} = (a_1, \dots, a_n)$ with $0 \leq a_i < i$, and $f_{\underline{a}} \in \mathcal{Z}_n$. Using Theorem 1.0.2 with $n = r$ we can also see that $\mathcal{P}_n$ is a free $\mathcal{P}_n^{S_r}$-module on basis $\{x_1^{a_1} \dots x_r^{a_r} \mid 0 \leq a_i < i \text{ for all } 1 \leq i \leq r\}$. Now note that the polynomials $f_{\underline{a}} x_{r+1}^{a_{r+1}} \dots x_n^{a_n}$ are in $\mathcal{P}_n^{S_r}$, so, since f is also in $\mathcal{P}_n^{S_r}$, it follows that $f_{\underline{a}} = 0$ in (1.3) unless $a_1 = \cdots = a_r = 0$. This completes the proof. □

2
Symmetric groups I

In order to illustrate the theory we are trying to develop let us start from an "easy" special case, namely the case of complex representations of the symmetric group S_n. We explain the beautiful elementary approach of Okounkov and Vershik [OV] (see also [DG]). The idea of this approach is not new: to study all symmetric groups at once. However, it is rather amazing that in this way the whole theory can be built quickly from scratch using only the classical Maschke and Wedderburn–Artin Theorems.

We will obtain the following well-known results: labeling the irreducible $\mathbb{C}S_n$-modules by partitions of n, construction of Young's orthogonal bases in irreducible modules, explicit description of matrices of simple transpositions with respect to these bases, and the Murnaghan–Nakayama formula for irreducible characters.

2.1 Gelfand–Zetlin bases

Define the kth *Jucys–Murphy elements* (JM-element for short) $L_k \in FS_n$ as follows:

$$L_k := \sum_{1 \leq m < k} (m, k). \tag{2.1}$$

These elements were introduced in [Ju], [Mu$_1$]. Note that $L_1 = 0$ and L_k commutes with S_{k-1}. As $L_k \in FS_k$, it follows that the JM-elements commute. Here and below, if $m < n$, the default embedding of S_m into S_n is with respect to the *first* m letters. A copy of S_m embedded with respect to the *last* m letters is denoted by S'_m.

Denote by $\mathcal{Z}_n$ the center of the group algebra FS_n. Also let

$$\mathcal{Z}_{n,m} := (FS_{n+m})^{S_n}$$

be the centralizer of FS_n in FS_{n+m}. It is clear that $\mathcal{Z}_{n,m}$ is spanned by the class sums corresponding to the S_n-conjugacy classes in S_{n+m}. These conjugacy classes can be thought of as cycle shapes with "fixed positions" for $n+1$, $n+2, \dots, n+m$ – we call them *marked cycle shapes*. For example, the symbol

$$(*, *, *, *, *, *)(*, *)(*)(*)(12, *, 13, 14, *)(15) \tag{2.2}$$

corresponds to the S_{11}-conjugacy class in S_{15}, which consists of all permutations whose cycle presentation is obtained by inserting the numbers 1 through 11 instead of asterisks.

Proposition 2.1.1 [O_1] *The algebra $\mathcal{Z}_{n,m}$ is generated by S'_m, $\mathcal{Z}_n$, and $L_{n+1}, \dots, L_{n+m}$.*

Proof It is clear that S'_m, $\mathcal{Z}_n$, and $L_{n+1}, \dots, L_{n+m}$ are contained in $\mathcal{Z}_{n,m}$, so they generate a subalgebra $\mathcal{A} \subseteq \mathcal{Z}_{n,m}$. Conversely, let us filter $\mathcal{Z}_{n,m}$ so that the ith filtered component $\mathcal{Z}^i_{n,m}$ is the span of the class sums which correspond to the marked cycle shapes moving at most i elements. For example the class sum corresponding to (2.2) belongs to $\mathcal{Z}^{12}_{11,4}$, but not $\mathcal{Z}^{11}_{11,4}$. We prove by induction on $i = 0, 1, \dots$ that $\mathcal{Z}^i_{n,m} \subseteq \mathcal{A}$. For $i = 0$ and 1, we have $\mathcal{Z}^i_{n,m} = F \cdot 1 \subseteq \mathcal{A}$. We explain the inductive step on example. Let $z \in \mathcal{Z}^{12}_{11,4}$ be the class sum corresponding to the marked cycle shape from (2.2). Let $c \in \mathcal{Z}_{11}$ denote the sum of all elements of S_{11} whose cycle shape is

$$(*, *, *, *, *)(*, *)(*)(*).$$

Also, let

$$x = (12, 13)L_{12}(13, 14)(L_{14} - (12, 14) - (13, 14)) \in \mathcal{A}.$$

(Note that L_{12} is the class sum corresponding to $(*, 12)$, and $(L_{14} - (12, 14) - (13, 14))$ is the class sum corresponding to $(*, 14)$.) Then xc is equal to z modulo lower layers of our filtration. □

From now on until the end of Chapter 2 we assume that $F = \mathbb{C}$.

The following key multiplicity-freeness result is well known – it is a special case of the branching rule, which describes the restriction of an irreducible $\mathbb{C}S_n$-module to S_{n-1}. However, usually the branching rule is proved after some theory has been developed and irreducible modules have been studied. In the approach explained here the multiplicity-freeness result is proved from scratch and then used to develop a theory.

Theorem 2.1.2 *Let V be an irreducible $\mathbb{C}S_n$-module. Then the restriction $\text{res}_{S_{n-1}} V$ is multiplicity free.*

Proof It follows from Proposition 2.1.1 that the centralizer of $\mathbb{C}S_{n-1}$ in $\mathbb{C}S_n$ is commutative. So the theorem comes from Lemmas 1.0.1. □

We now define the *branching graph* $\mathbb{B}$ whose vertices are isomorphism classes of irreducible $\mathbb{C}S_n$-modules for all $n \geq 0$ (by agreement $\mathbb{C}S_0 = \mathbb{C}$); we have a directed edge $W \to V$ from (an isoclass of) an irreducible $\mathbb{C}S_n$-module W to (an isoclass of) an irreducible $\mathbb{C}S_{n+1}$-module V if and only if W appears as a composition factor of $\text{res}_{S_n} V$; there are no other edges. Our main goal is to find an explicit combinatorial description of the branching graph. This will give us a labeling of the irreducible $\mathbb{C}S_n$-modules *for all* n. This will also yield the branching rule. To achieve this goal we will actually do more.

Let V be an irreducible $\mathbb{C}S_n$-module. Pick an S_n-invariant inner product $(\cdot, \cdot)$ on V (it is unique up to a scalar). Theorem 2.1.2 implies that the decomposition

$$\text{res}_{S_{n-1}} V = \bigoplus_{W \to V} W$$

is canonical. Decomposing each W on restriction to S_{n-2}, and continuing inductively all the way to S_0, we get a canonical decomposition

$$\text{res}_{S_0} V = \bigoplus_{T} V_T$$

into irreducible $\mathbb{C}S_0$-modules, that is 1-dimensional subspaces V_T, where T runs over all paths $W_0 \to W_1 \to \cdots \to W_n = V$ in $\mathbb{B}$. Note that

$$\mathbb{C}S_k \cdot V_T = W_k \qquad (0 \leq k \leq n). \tag{2.3}$$

Choosing a vector $v_T \in V_T$, we get a basis $\{v_T\}$ of V called *Gelfand–Zetlin basis* (or GZ-basis). Vectors of GZ-basis are defined uniquely up to scalars. Moreover, if $\varphi : V \to V'$ is an isomorphism of irreducible modules then φ moves a GZ-basis of V to a GZ-basis of V'. Note also, for example using (2.3), that a GZ-basis is orthogonal with respect to $(\cdot, \cdot)$.

Now decompose the algebra $\mathbb{C}S_n$ according to the Wedderburn–Artin Theorem

$$\mathbb{C}S_n = \bigoplus_{V} \text{End}_{\mathbb{C}}(V), \tag{2.4}$$

where the sum is over the representatives of the isoclasses of irreducible $\mathbb{C}S_n$-modules. This decomposition is canonical. Let us pick a GZ-basis in each V. Then we also identify

$$\mathbb{C}S_n = \bigoplus_V M_{\dim V}(\mathbb{C}). \tag{2.5}$$

Define the *GZ-subalgebra* $\mathcal{A}_n \subseteq \mathbb{C}S_n$ as the subalgebra which consists of all elements of $\mathbb{C}S_n$, which are diagonal with respect to a GZ-basis in every irreducible $\mathbb{C}S_n$-module. In terms of the decomposition (2.5), $\mathcal{A}_n$ consists of all diagonal matrices. In particular:

Lemma 2.1.3 *$\mathcal{A}_n$ is a maximal commutative subalgebra of $\mathbb{C}S_n$. Also, $\mathcal{A}_n$ is a semisimple algebra.*

We give two more descriptions of the GZ-subalgebra.

Lemma 2.1.4

(i) *$\mathcal{A}_n$ is generated by the subalgebras $\mathcal{Z}_0, \mathcal{Z}_1, \dots, \mathcal{Z}_n \subseteq \mathbb{C}S_n$.*
(ii) *$\mathcal{A}_n$ is generated by the JM-elements $L_1, L_2, \dots, L_n$.*

Proof (i) Let $e_V \in \mathcal{Z}_n$ be the central idempotent of $\mathbb{C}S_n$, which acts as identity on V and as zero on any irreducible $\mathbb{C}S_n$-module $V' \not\cong V$ (in terms of (2.4); e_V is the identity endomorphism in the V-component and zero endomorphism in other components). If $T = W_0 \to W_1 \to \cdots \to W_n = V$ is a path in $\mathbb{B}$ then

$$e_{W_0} e_{W_1} \dots e_{W_n} \in \mathcal{Z}_0 \mathcal{Z}_1 \dots \mathcal{Z}_n$$

acts as the projection to V_T along $\oplus_{S \neq T} V_S$ and as zero on any irreducible $\mathbb{C}S_n$-module $V' \not\cong V$. So the subalgebra generated by $\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_n$ contains $\mathcal{A}_n$. As this subalgebra is commutative and $\mathcal{A}_n$ is a maximal commutative subalgebra of $\mathbb{C}S_n$, the two must coincide.

(ii) Note that L_k is the sum of all transpositions in S_k minus the sum of all transpositions in S_{k-1}, that is L_k is a difference of a central element in S_k and a central element in S_{k-1}. So by (i), the JM-elements do belong to $\mathcal{A}_n$. To prove that they generate $\mathcal{A}_n$, proceed by induction on n, the inductive base being trivial. By (i), $\mathcal{A}_n$ is generated by $\mathcal{A}_{n-1}$ and $\mathcal{Z}_n$. In view of the inductive assumption, it suffices to prove that $\mathcal{A}_{n-1}$ and L_n generate $\mathcal{Z}_n$. But this follows from the obvious embedding $\mathcal{Z}_n \subseteq \mathcal{Z}_{n-1,1}$ and Proposition 2.1.1, as $\mathcal{Z}_{n-1} \subseteq \mathcal{A}_{n-1}$. □

Now, we will try to have the GZ-subalgebra play a role of a Cartan subalgebra in Lie Theory. As $\mathcal{A}_n$ is semisimple we can decompose every