The Direct Method in Soliton Theory

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Foreword

The second half of the twentieth century saw a resurgence in the study of classical physics. Scientists began paying particular attention to the effects caused by the nonlinearity in dynamical equations. This nonlinearity was found to have two interesting manifestations of opposite nature: *chaos*, that is the apparent randomness in the behaviour of perfectly deterministic systems, and *solitons*, that is localized, stable moving objects that scattered elastically. Both of these topics have now been developed into paradigms, with solid mathematical background and with a wide range of physical observations and concrete applications.

This book is concerned with a particular method used in the study of solitons. There are many ways of studying the integrable nonlinear evolution equations that have soliton solutions, each method having its own assumptions and areas of applicability. For example, the inverse scattering transform (IST) can be used to solve initial value problems, but it uses powerful analytical methods and therefore makes strong assumptions about the nonlinear equations. On the other hand, one can find a travelling wave solution to almost all equations by a simple substitution which reduces the equation to an ordinary differential equation. Between these two extremes lies Hirota's direct method. Although the transformation was, at its heart, inspired by the IST, Hirota's method does not need the same mathematical assumption and, as a consequence, the method is applicable to a wider class of equations than the IST. At the same time, because it does not use such sophisticated techniques, it usually produces a smaller class of solutions, the multi-soliton solutions. In many problems the key to further developments is a detailed understanding of soliton scattering, and in such cases Hirota's bilinear method is the optimal tool.

Over the years, many textbooks have been written on various aspects of solitons. Although some of them have briefly mentioned Hirota's bilinear method, there has not been any introductory English language book devoted to it. When some of the western practitioners of Hirota's method found out about prof. Hirota's book, they felt that it should be translated into English for use as an introduction to the method. Early in 1997, Prof. J. Satsuma recruited his students A. Nagai for this translation project. With help from his colleagues, S. Tsujimoto and R. Willox, a first version was made; this was further improved in a collaboration between A. Nagai, J. J. C. Nimmo and C. R. Gilson, leading to the final version that is presented here.

In this book, Prof. Hirota explains his 'direct' or 'bilinear' method. There is an interesting introduction, from which we can see the motivation and chain of thought that led Prof. Hirota to invent his method. The rest of the book is devoted to a detailed discussion of various applications of the method. Little has been changed in the translation from the original book; one or two arguments have been expanded, some errors have been corrected and some notation was changed to improve consistency. All such changes have been made with the approval of Prof. Hirota.

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Preface

A soliton is a particular type of solitary wave, which is not destroyed when it collides with another wave of the same kind. Such behaviour is suggested by numerical simulation, but is it really possible that the soliton completely recovers its original shape after a collision? In detailed analysis of the results of such numerical simulations, some ripples can be observed after a collision, and it therefore seems that the original shape is not completely recovered. Therefore, in order to clarify whether or not solitons are destroyed through their collisions, it is necessary to find exact solutions of soliton equations.

Generally, it is a very hard task to find exact solutions of nonlinear partial differential equations, including soliton equations. Moreover, even if one manages to find a method for solving one nonlinear equation, in general such a method will not be applicable to other equations. Does there exist any successful and universal tool enabling one to solve many types of nonlinear equations which does not require a deep understanding of mathematics? For this purpose, a direct method has been investigated.

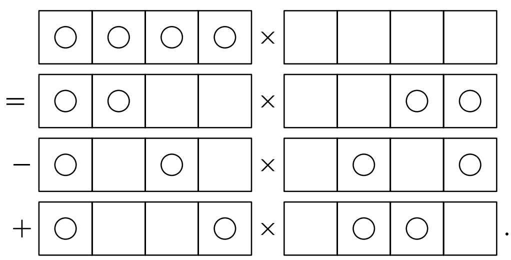
In Chapter 1, we discuss in an intuitive way the conditions under which a solitary wave is formed and we show that a nonlinear solitary wave cannot be made by the superposition of linear waves. From this observation, we obtain a method for finding solutions of a nonlinear wave equation by reductive perturbations and derive the fundamental idea for the direct method. We first introduce new dependent variables F and G to express the solution of the equation as the ratio G/F and then solve the (now bilinear) equations for F and G. As part of this, a new binary operator, called the D-operator, is derived. General formulae, through which nonlinear partial differential equations are transformed into bilinear (or, in general, homogeneous) forms, are presented. By virtue of special properties of the D-operator, solving these bilinear forms by ordinary reductive perturbation methods leads to perturbation expansions that may sometimes be truncated as finite sums. Such a truncation yields an exact solution for the equation. As an example, we find an exact solution for one of the most famous soliton equations, the KdV equation, and prove that its solitons are preserved after interaction.

In Chapter 2, we introduce the mathematical tools – in particular the theory of determinants and pfaffians – to be used in Chapter 3. These techniques will be explained thoroughly by means of several examples so that readers with only elementary knowledge can understand them. Consequently, this chapter covers one-quarter of the book.

In Chapter 3, we discuss the structure of soliton equations from the viewpoint of the direct method presented in this book. Many kinds of soliton equations have been discovered up to now and it would require several pages to write them all down. Now the question arises: what is the fundamental structure common to all soliton equations? The answer is provided in this chapter; soliton equations (or bilinear forms) are nothing but 'pfaffian identities'. From this viewpoint, we show how fundamental soliton equations, such as the KP, BKP, coupled KP, Toda lattice and Toda molecule equations, resolve themselves into pfaffian identities.

Pfaffians, which may be an unfamiliar word, are closely related to determinants. They are usually defined by the property that the square of a pfaffian is the determinant of an antisymmetric matrix. This property often gives rise to the misunderstanding that a pfaffian is merely a special case of a determinant. In fact, it is more natural to regard a pfaffian as a *generalization* of a determinant. For example, Plücker relations and Jacobi identities, which are identities for determinants, also hold for pfaffians. As a matter of fact, they can be extended and unified as pfaffian identities.

By means of the Maya diagrams designed by Professor Mikio Sato, a pfaffian identity can be illustrated by the formula



It is quite a surprise that soliton equations reduce to such simple diagrams!

In Chapter 4, we discuss Bäcklund transformations, which have made important contributions in the development of soliton theory. Bäcklund transformations in bilinear form generate (i) Lax pairs used in the inverse scattering method, (ii) new soliton equations, and (iii) Miura transformations. A Bäcklund transformation in bilinear forms corresponds to an 'exchange formula' for the D-operator. First, we find a Bäcklund transformation for the KdV equation by using such an exchange formula. Next, we illustrate some applications of this Bäcklund transformation for the KdV equation. Finally, we clarify the structure of Bäcklund transformations for other soliton equations such as the KP, BKP and Toda equations, and we also show that all these Bäcklund transformations reduce to pfaffian identities.

Since most of this book is devoted to an explanation of the fundamental facts concerning the direct method, space does not permit us to mention its applications in many other fields. It is particularly disappointing that we could not touch upon the group-theoretical aspects of bilinear forms developed by the Sato school (Professors Mikio Sato, Yasuko Sato, Masaki Kashiwara, Tetsuji Miwa, Michio Jimbo and Etsuro Date). With regard to inverse scattering methods, we have completely omitted them because many books have already been written on this subject. The aim of this book is to inform the readers as briefly as possible about the beauty and conciseness of the mathematical rules underlying soliton equations.

The author is greatly indebted to members of Professor Mikio Sato's school in Kyoto University and those of Professor Junkichi Satsuma's laboratory in the University of Tokyo, for their own developments of direct methods. He also thanks Dr Hideyuki Kidachi, whose notes on the author's lectures (Department of Physics, Faculty of Science, Kyoto University, 1–3 February 1979) were very useful in writing Chapter 1. Last but not least, the author is grateful to Mr Satoshi Tsujimoto and Mr Tatsuya Imai for their help drawing figures and proofreading.